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FLUTTER AND OSCILLATING AIR-FORCE CALCULATIONS FOR AN AIRFOIL IN A TWO-DIMENSIONAL SUPERSONIC FLOW

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SUMMARY

A connected account is given of the Possio theory of nonstationary flow for small disturbances in a two-dimensional supersonic flow and of its application to the determination of the aerodynamic forces on an oscillating airfoil. Further application is made to the problem of wing flutter in the degrees of freedom - torsion, bending, and aileron torsion. Numerical tables for flutter calculations are provided for various values of the Mach number greater than unity. Results for bending-torsion wing flutter are shown in figures and discussed. The static instabilities of divergence and aileron reversal are examined as is a one-degree-of-freedom case of torsional oscillatory instability.

INTRODUCTION

The problem of flutter or aerodynamic instability for high-speed aircraft is of considerable importance and hence interest is directed to the aerodynamic problem of the oscillating airfoil moving forward at high speed. Although for conventional aircraft the subsonic and the near-sonic or transonic speed ranges are still of main interest, the purely supersonic speed range is becoming increasingly significant.

A theoretical treatment of the oscillating airfoil, of infinite aspect ratio, moving at supersonic speed has been given by Possio (reference 1). This treatment is based on the theory of small perturbations to the main stream, thus is essentially an acoustic theory, and leads to linearization of the equation satisfied by the velocity potential. The airfoil is therefore assumed to be very thin, at small angle of attack, and the flow is assumed nonviscous, unseparated, and free from strong shocks.

The small-disturbance linearized theory, being much less complicated than a more rigorous nonlinear theory, is to be regarded as an expedient which allows an initial theoretical solution. The theory permits the occurrence of weak (infinitesimally small) shocks and thus the basic trends and effects of the parameters of the simplified problem can be indicated. The theory reduces to that of Ackeret in the stationary (static) case and, like it, is not expected to be valid too near $M = 1$. In view of the restrictions and assumptions in the analysis important modifications may be required in certain cases for thick finite airfoils, but even here the simple theory for thin wing sections may serve as a basis.

In addition to Possio's brief work an equivalent extended treatment has been given by Borbely (reference 2) which utilizes contour integrations to carry out the solution of the partial differential equation for the velocity potential according to the Heaviside operator method or Laplace transform method. Recently, another equivalent treatment has been given in England by Temple and Jahn employing the method of characteristics. In reference 1 a few curves are given for the aerodynamic coefficients but no numerical values are tabulated. Reference 2 contains no numerical results. Temple and Jahn recognize the lack of numerical results and supply some initial calculations for the functions necessary for flutter calculations.

A paper has recently appeared by Schwarz (reference 3) devoted to computing and tabulating the key mathematical functions that arise in the theory. The present paper makes use of reference 3 to supply more extensive numerical tables for application of the theory. The formulas of the theory are recast in more familiar form for application to the flutter problem and a series of calculations on bending-torsion flutter are carried out and discussed. The performance of similar calculations for wing-aileron flutter is indicated. Brief discussions also are given of the static instabilities, divergence and aileron reversal, and of a one-degree-of-freedom torsional oscillatory instability.

For completeness, a connected account of the Possio theory is presented since the original presentation in Italian is quite terse and also since it is believed that this treatment is the simplest and most suitable for general extensions. The extension of its application to include the aileron is given.

AIR FORCES AND MOMENTS ON AN OSCILLATING AIRFOIL MOVING AT SUPERSONIC SPEED IN TWO-DIMENSIONAL FLOW

Differential Equation for the Velocity Potential

The differential equation satisfied by the velocity potential in fixed coordinates in the case of infinitesimal disturbances is the wave equation

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = \nabla^2 \phi \tag{1}$$

where c is the velocity of sound in the undisturbed medium. (For the adiabatic equation of state $c^2 = \frac{dp}{d\rho} = \gamma \frac{p}{\rho}$.)

Referred to a system of rectangular coordinates moving forward at a constant supersonic speed v in the negative x-direction the wave equation satisfied by the velocity potential in two-dimensional flow becomes

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} + \frac{2v}{c^2}\frac{\partial^2 \phi}{\partial x\, \partial t} + \left[\left(\frac{v}{c}\right)^2 - 1\right]\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{2}$$

It is proposed to treat the effect of a slightly cambered thin airfoil moving forward at a supersonic speed v at small (zero) angle of attack as that of a distribution of small disturbances placed along the x-axis and hence to utilize equation (2). The velocity components in the x- and y-directions relative to the moving airfoil are, respectively,

$$v_x = \frac{\partial \phi}{\partial x}$$

and

$$v_y = \frac{\partial \phi}{\partial y}$$

which may be considered the additional components to the main stream due to the disturbance created by the presence of the airfoil. Relative to coordinates fixed in space the velocity components are $v + v_x$ and v_y.

Effect of a Source

Equation (2) is linear and solutions are therefore additive. An important particular solution of equation (2) having the property of a source pulse is

$$\phi_o = \frac{A(\xi, \eta, T)}{\sqrt{c^2(t - T)^2 - \left[x - \xi - v(t - T)\right]^2 - (y - \eta)^2}} \qquad (3)$$

This solution may be considered to give the effect at a point (x, y) at time t of a disturbance of magnitude A originating at a point (ξ, η) at an earlier time T. The potential ϕ_o is thus a retarded potential and the elapsed time at (x, y) since the creation of the disturbance is $\tau = t - T$.

Unlike the situation for a subsonic flow, for a supersonic flow the effect of the disturbance is propagated only downstream, that is, the point being influenced (x, y) is always considered to be aft of the point of disturbance (ξ, η). Equation (3) is thus valid in the angular region with vertex at (ξ, η) and bounded by two straight lines making the Mach angles $\pm\mu = \pm\sin^{-1}\frac{c}{v} = \pm\sin^{-1}\frac{1}{M}$ with respect to the x-axis. (See fig. 1.) Upstream from this angular region the value of ϕ_o is zero. It follows also that disturbances in the wake need not be considered and the solution to the boundary problem may be attempted by a distribution of potentials of the type ϕ_o taken along the projection of the airfoil on the x-axis.

A disturbance at (ξ, η) created at time T is first felt at a point (x, y) after a certain time τ_1 has elapsed. The point (x, y) penetrates the wave front of the disturbed region and because it is moving at a speed greater than that of the wave front it emerges from the disturbed region at a later time τ_2. Thus, the duration of this initial disturbance at (x, y) is $\tau_2 - \tau_1$. (See fig. 2.) The transition at (x, y) from a region of quiescence to a region of disturbance and vice versa is associated with the vanishing of the denominator in equation (3). The values of τ_1 and τ_2 for a disturbance created on the axis $\eta = 0$ are thus given by

$$\tau_{1,2} = \frac{M(x - \xi) \mp \sqrt{(x - \xi)^2 - y^2(M^2 - 1)}}{c(M^2 - 1)} \tag{4}$$

where the minus sign is associated with τ_1 and the plus sign with τ_2 and where $M = \frac{v}{c}$. It may also be observed that a negative quantity under the radical sign in equation (3) is to be interpreted as associated with an undisturbed region (that is, with $\phi = 0$).

Potential for a Distribution of Sources

The total effect at any point (x, y) is the sum of the effects of disturbances originating between the leading edge $\xi = 0$ and the intersection of the Mach line through (x, y) with the ξ-axis

$$\xi = \xi_1 = x - y\sqrt{M^2 - 1}$$

(since only disturbances created forward of the Mach angle region can affect (x, y); see fig. 3).

The total potential at (x, y) at any time t is thus given by

$$\phi(x, y, t) = \int_0^{\xi_1}\int_{\tau_1}^{\tau_2} \frac{A(\xi,\ 0,\ t - \tau)}{\sqrt{c^2\tau^2 - (x - \xi - v\tau)^2 - y^2}}\, d\tau\, d\xi$$

$$= \frac{1}{\sqrt{v^2 - c^2}} \int_0^{\xi_1}\int_{\tau_1}^{\tau_2} \frac{A(\xi,\ 0,\ t - \tau)}{\sqrt{(\tau - \tau_1)(\tau_2 - \tau)}}\, d\tau\, d\xi \qquad (5)$$

Boundary Condition and Strength of Distribution

The function $A(\xi,\ 0,\ t - \tau)$ giving the magnitude of the source distribution is now to be determined by the usual boundary condition of tangential flow along the airfoil. If the ordinate of any point of the mean line defining the airfoil is given as $y = y_m(x, t)$ the boundary condition may be written

$$\left(\frac{\partial\phi}{\partial y}\right)_{y=0} = w(x,\ t) = \frac{dy}{dt}$$

$$= v\frac{\partial y_m}{\partial x} + \frac{\partial y_m}{\partial t} \qquad (6)$$

where $w(x,\ t)$ thus represents the vertical velocity induced by the source distribution in order to realize tangential flow at the airfoil boundary. (In the stationary case - Ackeret treatment - the two surfaces of the airfoil may be considered as acting independently, which can also be done for the nonstationary case. However, for the purpose of obtaining the oscillating forces in the linear treatment it is sufficient to consider separately the upper and lower sides of only the mean line.)

The evaluation of $\frac{\partial \phi}{\partial y}$ as y approaches zero may be readily obtained by use of the variable θ instead of τ where $2\tau = (\tau_2 - \tau_1)\cos\theta + \tau_2 + \tau_1$. This substitution in equation (5) yields

$$\phi = \frac{1}{\sqrt{v^2 - c^2}} \int_0^{\xi_1} \int_0^{\pi} A\left(\xi,\ 0,\ t - \frac{\tau_2 + \tau_1}{2} - \frac{\tau_2 - \tau_1}{2}\cos\theta\right) d\theta\, d\xi$$

By differentiation with regard to y and with the aid of an integration by parts

$$\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{v^2 - c^2}} \frac{\partial \xi_1}{\partial y} \pi A\left(\xi_1,\ 0,\ t - \frac{My}{c\sqrt{M^2 - 1}}\right)$$

$$+ \frac{1}{\sqrt{v^2 - c^2}} \frac{y}{c\sqrt{M^2 - 1}} \int_0^{\xi_1} \int_0^{\pi} \frac{\partial^2 A}{\partial t^2} \sin^2\theta\, d\theta\, d\xi$$

Since $\xi_1 = x - y\sqrt{M^2 - 1}$, there results in the limit as y approaches zero on the positive side, the important relation

$$\left(\frac{\partial \phi}{\partial y}\right)_{y=+0} = -\frac{\pi}{c} A(x,\ 0,\ t)$$

or, briefly,

$$A(x,\ t) = -\frac{c}{\pi} w(x,\ t) \tag{7}$$

For y approaching 0 on the negative side an equal and opposite result is obtained and hence the distribution of singularities to be utilized to replace the airfoil is of the source-sink type. Thus ϕ is to be understood in the subsequent analysis to be prefixed by a $\pm$ sign, + for the upper side and − for the lower side.

The total potential for $y = 0$ may now be expressed by means of equations (5) and (7) as

$$\phi(x, t) = -\frac{1}{\pi} \frac{1}{\sqrt{M^2 - 1}} \int_0^x \int_{\tau_1}^{\tau_2} \frac{w(\xi, t - \tau)}{\sqrt{(\tau - \tau_1)(\tau_2 - \tau)}} d\tau \, d\xi \quad (8)$$

where, from equation (4) with $y = 0$,

$$\tau_1 = \frac{x - \xi}{c} \frac{1}{M + 1}$$

and

$$\tau_2 = \frac{x - \xi}{c} \frac{1}{M - 1}$$

Application to Oscillating Airfoil

The general result given by equation (8) may now be applied for definiteness to the case of an airfoil performing small sinusoidal oscillations in several degrees of freedom. Let the wing undergo the following motions: a motion due to displacement h (velocity $\dot{h}$) in a vertical direction; a torsional motion consisting of a turning about $x = x_0$ with instantaneous angle of attack α; a rotation of an aileron about its hinge at $x = x_1$ with instantaneous aileron angle β measured with respect to α. (See fig. 4.)

In accordance with equation (6) the vertical velocity at any point x of the airfoil situated at $0 \leqq x \leqq 2b$ (of chord 2b and leading edge at $x = 0$) is easily recognized to be

$$w(x, t) = -\left[\dot{h} + v\alpha + (x - x_0)\dot{\alpha} + v\beta + (x - x_1)\dot{\beta}\right] \quad (9)$$

where the β-terms are to be interpreted as zero for $x < x_1$ (and where the minus sign is introduced because the vertical velocity w is positive upwards whereas the terms within the brackets are positive downwards).

It is convenient in treating sinusoidal motion to utilize the complex notation

$$\left.\begin{aligned} h &= h_0 e^{i\omega t} \\ \alpha &= \alpha_0 e^{i\omega t} \\ \beta &= \beta_0 e^{i\omega t} \end{aligned}\right\} \tag{10}$$

where h_0, α_0, and β_0 are complex amplitudes and hence include phase angles.

Since the further analysis is concerned only with exponential time variations of the type given in equation (10), the function $w(\xi, t - \tau)$ occurring in equation (8) is of the form $w(\xi)e^{i\omega(t-\tau)}$, which may also be written for convenience as $w(\xi, t)e^{-i\omega\tau}$. The potential ϕ given by equation (8) may now be written as

$$\phi(x, t) = -\frac{1}{\sqrt{M^2 - 1}}\int_0^x w(\xi, t)\, I(\xi, x)\, d\xi \tag{11}$$

where

$$I(\xi, x) = \frac{1}{\pi}\int_{\tau_1}^{\tau_2} \frac{e^{-i\omega\tau}}{\sqrt{(\tau - \tau_1)(\tau_2 - \tau)}}\, d\tau$$

The integration with regard to τ may be readily performed by substitution of the variable θ where $2\tau = (\tau_2 - \tau_1)\cos\theta + \tau_2 + \tau_1$. Then

$$I(\xi, x) = \frac{1}{\pi}\, e^{-i\omega(\tau_2+\tau_1)/2}\int_0^{\pi} e^{-i\omega\cos\theta(\tau_2-\tau_1)/2}\, d\theta$$

With τ_1 and τ_2 replaced by their values as given for equation (8) and with the aid of the Bessel function relation

$$\frac{1}{\pi}\int_0^{\pi} e^{-i\lambda\cos\theta}\, d\theta = J_0(\lambda)$$

it is recognized that

$$I(\xi, x) = e^{-i\omega\frac{x-\xi}{c}\frac{M}{M^2-1}} J_0\left(\frac{x-\xi}{c}\frac{\omega}{M^2-1}\right) \tag{12}$$

Throughout the subsequent analysis it is convenient to employ the variables x and ξ in a new sense to mean nondimensional quantities obtained by dividing the old variables by the chord $2b$. The retaining of the symbols x and ξ for the nondimensional variables should lead to no confusion.

The potential ϕ of equation (11) is then

$$\phi(x,t) = \frac{2b}{\sqrt{M^2-1}}\int_0^x \left[v\alpha + \dot{h} + 2b\left(\xi - x_0\right)\dot{\alpha} + v\beta + 2b\left(\xi - x_1\right)\dot{\beta}\right] I(\xi,x)\, d\xi \tag{13}$$

where with the introduction of the important frequency parameters

$$k = \frac{\omega b}{v}$$

$$\bar{\omega} = \frac{2kM^2}{M^2-1}$$

the function $I(\xi, x)$ becomes

$$I(\xi, x) = e^{-i\bar{\omega}(x-\xi)} J_0\left[\frac{\bar{\omega}}{M}(x-\xi)\right] \tag{12'}$$

Thus, $I(\xi, x)$ is a function of the variable $x - \xi$ and of two parameters M and $\overline{\omega}$, or, alternatively, M and k.

It is desirable to express the potential ϕ as the sum of the separate effects due to position and motion of the airfoil associated with the individual terms in equation (13). Thus

$$\phi(x, t) = \phi_\alpha + \phi_{\dot{h}} + \phi_{\dot{\alpha}} + \phi_\beta + \phi_{\dot{\beta}} \tag{14}$$

where

$$\phi_\alpha = \frac{2b}{\sqrt{M^2 - 1}} v\alpha \int_0^x I(\xi, x)\, d\xi$$

$$\phi_{\dot{h}} = \frac{2b}{\sqrt{M^2 - 1}} \dot{h} \int_0^x I(\xi, x)\, d\xi$$

$$\phi_{\dot{\alpha}} = \frac{4b^2}{\sqrt{M^2 - 1}} \dot{\alpha} \int_0^x \left(\xi - x_0\right) I(\xi, x)\, d\xi$$

$$\phi_\beta = \frac{2b}{\sqrt{M^2 - 1}} v\beta \int_{x_1}^x I(\xi, x)\, d\xi$$

$$\phi_{\dot{\beta}} = \frac{4b^2}{\sqrt{M^2 - 1}} \dot{\beta} \int_{x_1}^x \left(\xi - x_1\right) I(\xi, x)\, d\xi$$

Forces and Moments

The basic pressure formula in the theory of small disturbances is

$$p = -\rho\frac{d\phi}{dt}$$

which in the present case of the moving airfoil may be expressed as

$$p = -\rho\left(\frac{\partial\phi}{\partial t} + v\frac{\partial\phi}{\partial x}\right)$$

where ρ is the density in the undisturbed medium. The local pressure difference on the airfoil surface between the upper and lower surfaces at any point x (nondimensional) is

$$p' = -2\rho\left(\frac{\partial\phi}{\partial t} + \frac{v}{2b}\frac{\partial\phi}{\partial x}\right) \tag{15}$$

The total force (positive downward) on the airfoil is

$$\begin{aligned} P &= 2b\int_0^1 p'\, dx \\ &= -2\rho v\int_0^1 \frac{\partial\phi}{\partial x}\, dx - 4\rho b\int_0^1 \dot{\phi}\, dx \end{aligned} \tag{16}$$

The moment (positive clockwise; fig. 4) on the entire airfoil about any point x_0 is

$$\begin{aligned} M_\alpha &= 4b^2\int_0^1 \left(x - x_0\right)p'\, dx \\ &= -4\rho b v\int_0^1 \frac{\partial\phi}{\partial x}\left(x - x_0\right)\, dx - 8\rho b^2\int_0^1 \dot{\phi}\left(x - x_0\right)\, dx \end{aligned} \tag{17}$$

Similarly, the moment (positive clockwise; fig. 4) on the aileron about the hinge point x_1 is

$$M_\beta = 4b^2\int_{x_1}^{1}\left(x - x_1\right)p' \, dx$$

$$= -4\rho bv\int_{x_1}^{1} \frac{\partial \phi}{\partial x}\left(x - x_1\right) \, dx - 8\rho b^2\int_{x_1}^{1}\dot{\phi}\left(x - x_1\right) \, dx \qquad (18)$$

In the further reduction of equations (16) to (18), with the potential ϕ replaced by its separated form given in equation (14), the following sets of integral evaluations are required:

$$\int_0^1 \frac{\partial \phi_\alpha}{\partial x} \, dx = \frac{2b}{\sqrt{M^2 - 1}} \, v\alpha r_1(M, k)$$

$$\int_0^1 \frac{\partial \phi_{\dot{\alpha}}}{\partial x} \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \, \dot{\alpha}\left[r_2(M, k) - x_0 r_1(M, k)\right]$$

$$\int_{x_1}^1 \frac{\partial \phi_\beta}{\partial x} \, dx = \frac{2b}{\sqrt{M^2 - 1}} \, v\beta t_1\left(M, k, x_1\right)$$

$$\int_{x_1}^1 \frac{\partial \phi_{\dot{\beta}}}{\partial x} \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \, \dot{\beta} t_2\left(M, k, x_1\right)$$

$$\int_0^1 \phi_\alpha \, dx = \frac{2b}{\sqrt{M^2 - 1}} \, v\alpha r_2(M, k)$$

$$\int_0^1 \phi_{\dot{\alpha}} \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \, \dot{\alpha}\left[\frac{1}{2} r_3(M, k) - x_0 r_2(M, k)\right]$$

$$\int_{x_1}^{1} \phi_\beta \, dx = \frac{2b}{\sqrt{M^2 - 1}} v\beta \; t_2\left(M, k, x_1\right)$$

$$\int_{x_1}^{1} \phi_{\dot{\beta}} \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \dot{\beta} \; \frac{1}{2} \; t_3\left(M, k, x_1\right)$$

$$\int_{0}^{1} \frac{\partial \phi_\alpha}{\partial x} x \, dx = \frac{2b}{\sqrt{M^2 - 1}} v\alpha \; q_1(M, k)$$

$$\int_{0}^{1} \frac{\partial \phi_{\dot{\alpha}}}{\partial x} x \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \dot{\alpha} \left[\frac{1}{2} \; q_2(M, k) - x_0 \; q_1(M, k)\right]$$

$$\int_{x_1}^{1} \frac{\partial \phi_\beta}{\partial x} x \, dx = \frac{2b}{\sqrt{M^2 - 1}} v\beta \left[s_1\left(M, k, x_1\right) + x_1 \; t_1\left(M, k, x_1\right)\right]$$

$$\int_{x_1}^{1} \frac{\partial \phi_{\dot{\beta}}}{\partial x} x \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \dot{\beta} \left[\frac{1}{2} \; s_2\left(M, k, x_1\right) + x_1 \; t_2\left(M, k, x_1\right)\right]$$

$$\int_{0}^{1} \phi_\alpha x \, dx = \frac{2b}{\sqrt{M^2 - 1}} v\alpha \; \frac{1}{2} \; q_2(M, k)$$

$$\int_{0}^{1} \phi_{\dot{\alpha}} x \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \dot{\alpha} \left[\frac{1}{6} \; q_3(M, k) - \frac{1}{2} \; x_0 \; q_2(M, k)\right]$$

$$\int_{x_1}^{1} \phi_{\beta} x \, dx = \frac{2b}{\sqrt{M^2 - 1}} v\beta \left[\frac{1}{2} s_2(M, k, x_1) + x_1 \, t_2(M, k, x_1)\right]$$

$$\int_{x_1}^{1} \phi_{\dot{\beta}} x \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \dot{\beta} \left[\frac{1}{3} s_3(M, k, x_1) + \frac{1}{2} x_1 \, t_3(M, k, x_1)\right]$$

$$\int_{x_1}^{1} \frac{\partial \phi_{\alpha}}{\partial x} (x - x_1) \, dx = \frac{2b}{\sqrt{M^2 - 1}} v\alpha \, p_1(M, k, x_1)$$

$$\int_{x_1}^{1} \frac{\partial \phi_{\dot{\alpha}}}{\partial x} (x - x_1) \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \dot{\alpha} \left[\frac{1}{2} p_2(M, k, x_1) - x_0 \, p_1(M, k, x_1)\right]$$

$$\int_{x_1}^{1} \frac{\partial \phi_{\beta}}{\partial x} (x - x_1) \, dx = \frac{2b}{\sqrt{M^2 - 1}} v\beta \, s_1(M, k, x_1)$$

$$\int_{x_1}^{1} \frac{\partial \phi_{\dot{\beta}}}{\partial x} (x - x_1) \, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \dot{\beta} \, \frac{1}{2} s_2(M, k, x_1)$$

$$\int_{x_1}^{1} \phi_\alpha(x - x_1)\, dx = \frac{2b}{\sqrt{M^2 - 1}} v\alpha \frac{1}{2} p_2(M, k, x_1)$$

$$\int_{x_1}^{1} \phi_{\dot{\alpha}}(x - x_1)\, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \dot{\alpha} \left[\frac{1}{6} p_3(M, k, x_1) - \frac{1}{2} x_0 p_2(M, k, x_1)\right]$$

$$\int_{x_1}^{1} \phi_\beta(x - x_1)\, dx = \frac{2b}{\sqrt{M^2 - 1}} v\beta \frac{1}{2} s_2(M, k, x_1)$$

$$\int_{x_1}^{1} \phi_{\dot{\beta}}(x - x_1)\, dx = \frac{4b^2}{\sqrt{M^2 - 1}} \dot{\beta} \frac{1}{6} s_3(M, k, x_1)$$

The functions defined by the foregoing integral evaluations are further discussed in the following section; first, however, the force and moments (equations (16) to (18)) are given in their final forms as

$$P = -\frac{4\rho b}{\sqrt{M^2 - 1}}\left[v\left(v\alpha + \dot{h} - 2bx_0\dot{\alpha}\right)r_1 + 2b\left(2v\dot{\alpha} + \ddot{h} - 2bx_0\ddot{\alpha}\right)r_2 + 4b^2\ddot{\alpha}\frac{r_3}{2} + v^2\beta t_1 + 4bv\dot{\beta}t_2 + 4b^2\ddot{\beta}\frac{t_3}{2}\right] \tag{16'}$$

$$M_\alpha = -\frac{8\rho b^2}{\sqrt{M^2 - 1}}\left[v\left(v\alpha + \dot{h} - 2bx_0\dot{\alpha}\right)q_1 + 2b\left(2v\dot{\alpha} + \ddot{h} - 2bx_0\ddot{\alpha}\right)\frac{q_2}{2} + 4b^2\ddot{\alpha}\frac{q_3}{6} + v^2\beta(s_1 + x_1t_1) + 4bv\dot{\beta}\left(\frac{s_2}{2} + x_1t_2\right) + 4b^2\ddot{\beta}\left(\frac{s_3}{6} + x_1\frac{t_3}{2}\right)\right] - 2bx_0P \tag{17'}$$

$$M_\beta = -\frac{8\rho b^2}{\sqrt{M^2 - 1}}\left[v\left(v\alpha + \dot{h} - 2bx_0\dot{\alpha}\right)p_1 + 2b\left(2v\dot{\alpha} + \ddot{h} - 2bx_0\ddot{\alpha}\right)\frac{p_2}{2}\right.$$
$$\left. + 4b^2\ddot{\alpha}\,\frac{p_3}{6} + v^2\beta s_1 + 4bv\dot{\beta}\,\frac{s_2}{2} + 4b^2\ddot{\beta}\,\frac{s_3}{6}\right] \qquad (18')$$

Reduction and Evaluation of Foregoing Integrals

It is convenient to introduce the substitution $u = x - \xi$ and to express the function $I(\xi, x)$ (equation (12')) as

$$I(\xi, x) = I(u) = e^{-i\bar{\omega}u}\, J_0\left(\frac{\bar{\omega}}{M}\, u\right) \qquad (19)$$

The various functions defined by the foregoing sets of integrals may now be expressed as follows:

$$r_1(M, k) = \int_0^1 I(u)\, du$$

$$r_2(M, k) = \int_0^1 \int_0^x I(u)\, du\, dx$$

$$r_3(M, k) = 2\int_0^1 \int_0^x (x - u)\, I(u)\, du\, dx$$

$$q_1(M, k) = \int_0^1 u\, I(u)\, du$$

$$q_2(M, k) = 2\int_0^1 \int_0^x x\, I(u)\, du\, dx$$

$$q_3(M, k) = 6\int_0^1 \int_0^x x(x - u)\, I(u)\, du\, dx$$

$$p_1(M,\ k,\ x_1) = \int_{x_1}^{1} (u - x_1)\ I(u)\ du$$

$$p_2(M,\ k,\ x_1) = 2\int_{x_1}^{1} \int_{0}^{x} (x - x_1)\ I(u)\ du\ dx$$

$$p_3(M,\ k,\ x_1) = 6\int_{x_1}^{1} \int_{0}^{x} (x - x_1)(x - u)\ I(u)\ du\ dx$$

$$t_1(M,\ k,\ x_1) = \int_{0}^{1-x_1} I(u)\ du$$

$$t_2(M,\ k,\ x_1) = \int_{0}^{1-x_1} \int_{0}^{x} I(u)\ du\ dx$$

$$t_3(M,\ k,\ x_1) = 2\int_{0}^{1-x_1} \int_{0}^{x} (x - u)\ I(u)\ du\ dx$$

$$s_1(M,\ k,\ x_1) = \int_{0}^{1-x_1} u\ I(u)\ du$$

$$s_2(M,\ k,\ x_1) = 2\int_{0}^{1-x_1} \int_{0}^{x} x\ I(u)\ du\ dx$$

$$s_3(M,\ k,\ x_1) = 6\int_{0}^{1-x_1} \int_{0}^{x} x(x - u)\ I(u)\ du\ dx$$

Borbely (reference 2) has shown by means of reduction formulas that the six r- and q-functions may be obtained from a single integral. In a similar manner it may be indicated how the foregoing 15 functions may be obtained from the evaluation of the same integral. The reduction is accomplished in two stages. First, consider integrals of the following type:

$$\left.\begin{aligned} f_\lambda &= f_\lambda(M, \overline{\omega}) = \int_0^1 I(u)\ u^\lambda\ du \\ g_\lambda &= f_\lambda\left(M, \overline{\omega}x_1\right) = \frac{1}{x_1^{\lambda+1}} \int_0^{x_1} I(u)\ u^\lambda\ du \\ h_\lambda &= f_\lambda\left[M, \overline{\omega}\left(1 - x_1\right)\right] = \frac{1}{\left(1 - x_1\right)^{\lambda+1}} \int_0^{1-x_1} I(u)\ u^\lambda\ du \end{aligned}\right\} \quad (20)$$

By integration by parts it can be readily verified that the following relations hold

$$r_1 = f_0$$
$$r_2 = f_0 - f_1$$
$$r_3 = f_0 - 2f_1 + f_2$$

$$q_1 = f_1$$
$$q_2 = f_0 - f_2$$
$$q_3 = 2f_0 - 3f_1 + f_3$$

$$p_1 = q_1 - x_1 r_1 + x_1{}^2\left(g_0 - g_1\right)$$

$$p_2 = q_2 - 2x_1 r_2 + x_1{}^3\left(g_0 - 2g_1 + g_2\right)$$

$$p_3 = q_3 - 3x_1 r_3 + x_1{}^4\left(g_0 - 3g_1 + 3g_2 - g_3\right)$$

$$t_1 = \left(1 - x_1\right)h_0$$

$$t_2 = \left(1 - x_1\right)^2\left(h_0 - h_1\right)$$

$$t_3 = \left(1 - x_1\right)^3\left(h_0 - 2h_1 + h_2\right)$$

$$s_1 = \left(1 - x_1\right)^2 h_1$$

$$s_2 = \left(1 - x_1\right)^3\left(h_0 - h_2\right)$$

$$s_3 = \left(1 - x_1\right)^4\left(2h_0 - 3h_1 + h_3\right)$$

The final stage in the reduction of these functions is to utilize the following recursion formula (reference 2) obtained by integration by parts:

$$\begin{aligned}\frac{M^2 - 1}{M^2}\,\overline{\omega} f_\lambda(M, \overline{\omega}) = {} & \left[i + (1 - \lambda)\frac{1}{\overline{\omega}}\right]e^{-i\overline{\omega}}J_0\left(\frac{\overline{\omega}}{M}\right) - \frac{1}{M}\,e^{-i\overline{\omega}}J_1\left(\frac{\overline{\omega}}{M}\right) \\ & + i(1 - 2\lambda)\,f_{\lambda-1}(M, \overline{\omega}) \\ & + (1 - \lambda)^2\,\frac{1}{\overline{\omega}}\,f_{\lambda-2}(M, \overline{\omega}) \qquad (21)\end{aligned}$$

where $\lambda \geqq 1$ and f with a negative subscript is to be interpreted as zero.

The function $f_\lambda(M, \overline{\omega})$ may clearly refer also to the foregoing g- and h-functions, if $\overline{\omega}$ is replaced by the appropriate parameter; namely, $\overline{\omega}x_0$ for g_λ and $\overline{\omega}(1 - x_0)$ for h_λ. (See equations (20).) The recursion relation (equation (21)) thus reduces the various functions to the single function

$$f_0(M, \overline{\omega}) = \frac{1}{\overline{\omega}} \int_0^{\overline{\omega}} e^{-iu} J_0\left(\frac{u}{M}\right) du \tag{22}$$

which is therefore the only integral needed in the evaluation of the forces and moments.

The important integral in equation (22) has been recently made the subject of a mathematical investigation by Schwarz (reference 4). Schwarz gives tables of the values of its real and imaginary parts to eight decimal places for $0 \leqq \overline{\omega} \leqq 5$ and for $1 \leqq M \leqq 10$ for conveniently small intervals. For values of $\overline{\omega} > 5$ not given in Schwarz' tables, the function f_0 may be evaluated by means of the following series development (reference 2):

$$f_0(M, \overline{\omega}) = e^{-i\overline{\omega}} \sum_{n=0}^{\infty} \left(\frac{M^2 - 1}{M^2}\overline{\omega}\right)^n \frac{1}{2^n n!(2n+1)} \left[J_n(\overline{\omega}) + iJ_{n+1}(\overline{\omega})\right] \tag{23}$$

Table I gives values of the functions $f_0(M, \overline{\omega})$ based on the tables of Schwarz and on equation (23) for selected values of the Mach number $M = \frac{10}{9}, \frac{5}{4}, \frac{10}{7}, \frac{5}{3}, 2, \frac{5}{2}, \frac{10}{3}$, and 5 and for various appropriate values of $\overline{\omega}$ $\left(\text{or } \frac{1}{k}\right)$. Later use is made of the values given in table I for obtaining tables for flutter calculations.

EQUATIONS OF MOTION AND DETERMINANTAL EQUATION FOR FLUTTER CONDITION

The equations of motion and the border-line condition of unstable equilibrium yielding the flutter speed and frequency may be obtained exactly as in the incompressible case treated, for example, in reference 4. The two-dimensional treatment (infinite aspect ratio) is retained herein. Modifications due to assumed vibration modes of the finite wing may of course be introduced as in current practice (for example, reference 5). The modification of the forces and moments due to the three-dimensional nature of the flow is a more difficult problem which remains to be studied.

The equilibrium of the vertical forces, of the moments about the torsional axis $x = x_0$, and of the moments on the aileron about its hinge $x = x_1$ yields the three equations,

$$\left.\begin{aligned}
&\ddot{h}M + \ddot{\alpha}S_\alpha + \ddot{\beta}S_\beta + hC_h = P \\
&\ddot{\alpha}I_\alpha + \ddot{\beta}\left[I_\beta + 2b\left(x_1 - x_0\right)S_\beta\right] + \ddot{h}S_\alpha + \alpha C_\alpha = M_\alpha \\
&\ddot{\beta}I_\beta + \ddot{\alpha}\left[I_\beta + 2b\left(x_1 - x_0\right)S_\beta\right] + \ddot{h}S_\beta + \beta C_\beta = M_\beta
\end{aligned}\right\} \qquad (24)$$

where the various parameters are defined in the list of notation. (See appendix.)

In order to define the border-line condition of unstable equilibrium separating damped and undamped oscillations, the variables h, α, and β are used in the sinusoidal exponential form given in equation (10). For the desired condition, it is necessary that the equations (24) have a (nontrivial) solution for the complex amplitudes h_0, α_0, and β_0, or that the following determinantal equation hold:

$$\begin{vmatrix} \bar{A}_{ch} & A_{c\alpha} & A_{c\beta} \\ A_{ah} & \bar{A}_{a\alpha} & A_{a\beta} \\ A_{bh} & A_{b\alpha} & \bar{A}_{b\beta} \end{vmatrix} = 0 \tag{25}$$

where the complex elements of the determinant in separated form are

$$\bar{A}_{ch} = \Omega_h X - \mu + L_1 + iL_2$$

$$A_{c\alpha} = -\mu x_\alpha + L_3 + iL_4$$

$$A_{c\beta} = -\mu x_\beta + L_5 + iL_6$$

$$A_{ah} = -\mu x_\alpha + M_1 + iM_2$$

$$\bar{A}_{a\alpha} = \Omega_\alpha X - \mu r_\alpha{}^2 + M_3 + iM_4$$

$$A_{a\beta} = -\mu \left[r_\beta{}^2 + 2(x_1 - x_0)x_\beta\right] + M_5 + iM_6$$

$$A_{bh} = -\mu x_\beta + N_1 + iN_2$$

$$A_{b\alpha} = -\mu \left[r_\beta{}^2 + 2(x_1 - x_0)x_\beta\right] + N_3 + iN_4$$

$$\bar{A}_{b\beta} = \Omega_\beta X - \mu r_\beta{}^2 + N_5 + iN_6$$

and where the L's, M's, and N's are defined by the force and moment equations (16'), (17'), and (18') expressed in the following forms:

$$\left.\begin{aligned}
P &= -4\rho b\, v^2k^2e^{i\omega t}\left[\left(\frac{h_0}{b}\right)\left(L_1 + iL_2\right) + \alpha_0\left(L_3 + iL_4\right) + \beta_0\left(L_5 + iL_6\right)\right] \\
M_\alpha &= -4\rho b^2v^2k^2e^{i\omega t}\left[\left(\frac{h_0}{b}\right)\left(M_1 + iM_2\right) + \alpha_0\left(M_3 + iM_4\right) + \beta_0\left(M_5 + iM_6\right)\right] \\
M_\beta &= -4\rho b^2v^2k^2e^{i\omega t}\left[\left(\frac{h_0}{b}\right)\left(N_1 + iN_2\right) + \alpha_0\left(N_3 + iN_4\right) + \beta_0\left(N_5 + iN_6\right)\right]
\end{aligned}\right\} \quad (26)$$

Hence,

$$L_1 + iL_2 = \frac{1}{\sqrt{M^2 - 1}}\left(-2r_2 + \frac{i}{k}r_1\right)$$

$$L_3 + iL_4 = \frac{1}{\sqrt{M^2 - 1}}\left[-2r_3 + \frac{2i}{k}r_2 - \frac{i}{k}\left(-2r_2 + \frac{i}{k}r_1\right) - 2x_0\left(-2r_2 + \frac{i}{k}r_1\right)\right]$$

$$L_5 + iL_6 = \frac{1}{\sqrt{M^2 - 1}}\left[-2t_3 + \frac{2i}{k}t_2 - \frac{i}{k}\left(-2t_2 + \frac{i}{k}t_1\right)\right]$$

$$M_1 + iM_2 = \frac{1}{\sqrt{M^2 - 1}}\left[-2q_2 + \frac{2i}{k}q_1 - 2x_0\left(-2r_2 + \frac{i}{k}r_1\right)\right]$$

$$M_3 + iM_4 = \frac{1}{\sqrt{M^2 - 1}}\left\{-\frac{4}{3}q_3 + \frac{2i}{k}q_2 - \frac{i}{k}\left(-2q_2 + \frac{2i}{k}q_1\right) - 2x_0\left[-2r_3 + \frac{2i}{k}r_2 - \frac{i}{k}\left(-2r_2 + \frac{i}{k}r_1\right) - 2q_2 + \frac{2i}{k}q_1 - 2x_0\left(-2r_2 + \frac{i}{k}r_1\right)\right]\right\}$$

$$M_5 + iM_6 = \frac{1}{\sqrt{M^2 - 1}}\left\{-\frac{4}{3}s_3 + \frac{2i}{k}s_2 - \frac{i}{k}\left(-2s_2 + \frac{2i}{k}s_1\right) + 2\left(x_1 - x_0\right)\left[-2t_3 + \frac{2i}{k}t_2 - \frac{i}{k}\left(-2t_2 + \frac{i}{k}t_1\right)\right]\right\}$$

$$N_1 + iN_2 = \frac{1}{\sqrt{M^2 - 1}}\left(-2p_2 + \frac{2i}{k} p_1\right)$$

$$N_3 + iN_4 = \frac{1}{\sqrt{M^2 - 1}}\left[-\frac{4}{3} p_3 + \frac{2i}{k} p_2 - \frac{i}{k}\left(-2p_2 + \frac{2i}{k} p_1\right)\right.$$

$$\left. -2x_0\left(-2p_2 + \frac{2i}{k} p_1\right)\right]$$

$$N_5 + iN_6 = \frac{1}{\sqrt{M^2 - 1}}\left[-\frac{4}{3} s_3 + \frac{2i}{k} s_2 - \frac{i}{k}\left(-2s_2 + \frac{2i}{k} s_1\right)\right]$$

The determinantal equation (25) with the foregoing complex elements is equivalent to two real simultaneous equations and hence may be solved for two unknowns. In a given case the usual unknowns are the flutter speed v and the flutter frequency ω or, more conveniently, the related nondimensional parameters X and $1/k$. The parameter X appears linearly and only in the major diagonal elements (with bars), while the parameter $1/k$ appears transcendentally in every element of the determinant. Hence an obvious procedure though not the simplest for obtaining the simultaneous solutions of the two equations is to fix values of $1/k$, to solve for the roots of the two polynomials in X, to plot graphically these roots against $1/k$, and to note the points of intersection.

In a systematic numerical study of flutter any two parameters may be utilized as unknowns instead of X and $1/k$, which is often more convenient. A discussion of such procedure and the use of a method of elimination for simplifying the calculations is given in the appendix of reference 6.

The application to the two-degree-of-freedom subcase of bending-torsion flutter is treated more fully in the following section.

APPLICATION TO BENDING-TORSION FLUTTER

The determinantal equation in the two degrees of freedom h and α is

$$\begin{vmatrix} \bar{A}_{ch} & A_{c\alpha} \\ A_{ah} & \bar{A}_{a\alpha} \end{vmatrix} = 0$$

or

$$\begin{vmatrix} \Omega_h X - \mu + L_1 + iL_2 & -\mu x_\alpha + L_3 + iL_4 \\ -\mu x_\alpha + M_1 + iM_2 & \Omega_\alpha X - \mu r_\alpha^2 + M_3 + iM_4 \end{vmatrix} = 0 \quad (27)$$

The two equations in X obtained by equating the real and imaginary parts separately to zero are

$$\left.\begin{aligned} &\Omega_h\Omega_\alpha X^2 + \left[\Omega_\alpha\left(L_1 - \mu\right) + \Omega_h\left(M_3 - \mu r_\alpha^2\right)\right]X + C_R = 0 \\ &\text{and} \\ &\qquad\left(\Omega_\alpha L_2 + \Omega_h M_4\right)X + C_I = 0 \end{aligned}\right\} \quad (27')$$

where

$$C_R = \mu\left[x_\alpha\left(M_1 + L_3\right) - \left(M_3 - \mu r_\alpha^2\right) - L_1 r_\alpha^2 - \mu x_\alpha^2\right] + D_R$$

and

$$C_I = \mu\left[x_\alpha\left(M_2 + L_4\right) - M_4 - L_2 r_\alpha^2\right] + D_I$$

where

$$D_R = L_1M_3 - L_3M_1 - L_2M_4 + L_4M_2$$

and

$$D_I = L_1M_4 - L_4M_1 + L_2M_3 - L_3M_2$$

For convenience in numerical tabulation, it is desirable to introduce primed quantities, independent of the parameter x_0, defined by the following relations:

$$\left.\begin{aligned}
L_3 &= L_3' - 2x_0L_1 \\
L_4 &= L_4' - 2x_0L_2 \\
M_1 &= M_1' - 2x_0L_1 \\
M_2 &= M_2' - 2x_0L_2 \\
M_3 &= M_3' - 2x_0\left[(M_1' + L_3') - 2x_0L_1\right] \\
M_4 &= M_4' - 2x_0\left[(M_2' + L_4') - 2x_0L_2\right]
\end{aligned}\right\} \tag{28}$$

In table II convenient expressions for the quantities L_1, L_2, L_3', L_4', M_1', M_2', M_3', and M_4' are given and tabulated together with the combinations $M_1' + L_3'$ and $M_2' + L_4'$. Clearly these quantities depend on the function f_0 given in table I and hence the tabulation is made for the same values of M and $1/k$ (or ϖ). In addition, table II contains values for the quantities D_R and D_I which, in fact, are independent of x_0 and may be expressed as

$$D_R = L_1M_3' - L_3'M_1' - L_2M_4' + L_4'M_2'$$

and

$$D_I = L_1M_4' - L_4'M_1' + L_2M_3' - L_3'M_2'$$

The numerical application in the case of bending-torsion flutter has been performed for various selected examples. In most of the calculations the numerical procedure was to fix values of $1/k$, eliminate X, and solve for the parameter x_α. Interpolation was also used to obtain additional points in order to improve the fairing of some of the curves. Values of $1/k$ less than 1 did not yield any flutter points in this procedure. Results are shown plotted in a number of figures (figs. 5 to 20); however, before these figures are discussed, it is desirable to explain the significance of the parameters and the numerical values assigned to them.

The parameter μ may be considered to signify the wing density and three selected values 3.927, 7.854, and 15.708 in the order of increasing wing density have been mainly used in the calculations. (These values correspond to values of $\frac{1}{\kappa} = 5$, 10, and 20 in the notation of reference 4.) Alternatively, an increase in μ may be interpreted as an increase in altitude for a fixed wing density. The parameter μ may be expected to range up to high values for actual supersonic wings at high altitude. Only a few calculations, however, have been made for high values of μ ($\mu = 78.54$, $\frac{1}{\kappa} = 100$; see fig. 18).

The parameter ω_h/ω_α is the ratio of the wing bending frequency to the wing torsional frequency and may be expected normally to be less than unity. The three values 0, 0.707, and 1 have been largely used in the calculations although other values up to 2 have also been studied.

The parameter x_0 represents the position of the elastic axis measured from the leading edge and the three values 0.4, 0.5, and 0.6 represent, respectively, positions at 40, 50, and 60 percent chord. (These values correspond to values of $a = -0.2$, 0, and 0.2 in the notation of reference 4.)

The parameter x_α represents the distance of the center of gravity from the elastic axis. For example, $x_\alpha = 0.2$ represents a position of the center of gravity 10 percent of the chord behind the elastic axis. In many of the calculations x_α has been regarded as variable.

The parameter r_α^2 represents the radius of gyration of the wing about the elastic axis and has been kept fixed at the value $r_\alpha^2 = 0.25$.

The ordinate in figures 5 to 20 is the nondimensional flutter coefficient $v/b\omega_\alpha$ where $b\omega_\alpha$ is a convenient reference speed. This coefficient is also a function of the Mach number $M = \frac{v}{c}$ and several values of M have been employed in the calculations.

In a plot of the flutter coefficient $v/b\omega_\alpha$ against M, straight lines drawn from the origin at angle δ and intersecting the curves may be given an interesting interpretation (fig. 17). The slope of the line is given by $\frac{v/b\omega_\alpha}{v/c} = \frac{c}{b\omega_\alpha}$ or $\cot \delta = \frac{b\omega_\alpha}{c}$. Thus, $\cot \delta$ is directly proportional to the product of the chord and the torsional frequency. The question of whether at a given value of M the value of $b\omega_\alpha$ which will just prevent flutter is also sufficient to prevent flutter at neighboring higher values of M is answered by the simple criterion of whether $\cot \delta$ increases or decreases. In figure 17 two typical flutter curves are shown. In curve B the value of $b\omega_\alpha$ just necessary to prevent flutter at a speed corresponding to the value of M at P_2 is insufficient to prevent flutter at any higher value of M for which the flutter curve is below the straight line OP_2. For the type of curve A a maximum value of δ occurs at the "design critical points" P_1. The values of $b\omega_\alpha$ just necessary to prevent flutter at a speed corresponding to the value of M at P_1 is also sufficient to prevent flutter at all higher speeds.

The following salient facts may be extracted by inspection of the figures. Flutter exists or is possible for various ranges of the parameters but, in general, compared with subsonic cases the ranges of the parameters yielding flutter are more restricted.

The chordwise position of the aerodynamic center, the center of the oscillating pressure, is an important factor in the consideration of flutter. In the simple theory the midchord is the aerodynamic center for $M \gg 1$. For subsonic speeds, $M \ll 1$, the linearized theory indicates the quarter-chord position as the aerodynamic center. It should be expected that in the transonic region near $M = 1$ the aerodynamic center may shift considerably. From this point of view alone conclusions drawn from the simple theory for the range near $M = 1$ may require large modifications. The nature of the modifications may be roughly inferred by further experimental and theoretical study of the behavior of center-of-pressure locations.

For low values of the ratio of bending frequency to torsional frequency $\dfrac{\omega_h}{\omega_\alpha} \approx 0$ the position of the center of gravity relative to the aerodynamic center is important. For center-of-gravity positions forward of the midchord no flutter exists, whereas for positions behind the midchord there is a sharp decrease in the flutter coefficient from infinity; the position of the elastic axis influences the value of the flutter coefficient in this region, forward positions being more favorable (figs. 5(a) to 16(a)).

For values of $\dfrac{\omega_h}{\omega_\alpha} \approx 1$ the position of the center of gravity relative to the elastic axis becomes of more importance. For center-of-gravity positions forward of the elastic axis no flutter exists, whereas for positions behind the elastic axis flutter does occur, and a relative minimum coefficient appears for center-of-gravity positions only slightly (a few percent of the chord) behind the elastic axis.

The intermediate case, for which $\dfrac{\omega_h}{\omega_\alpha} = 0.707$, shows a blending of the effects in which the center-of-gravity position relative both to the aerodynamic center and to the elastic axis is significant.

In figures 12 and 14 there are shown, for reference, some numerical values of ω/ω_α, the ratio of the flutter frequency to the torsional frequency.

The effect of the wing-density parameter μ is rather complicated but, in general, an increase in μ yields a corresponding increase in the flutter coefficient. For low values of ω_h/ω_α and for high wing densities this increase is expected to be proportional to $\sqrt{\mu}$. In the resonance-like region near $\frac{\omega_h}{\omega_\alpha} = 1$ and for small values of x_α the flutter coefficient is relatively unaffected by the value of μ, and in this region the structural damping may be expected to be particularly effective in increasing the flutter coefficient.

For values of the Mach number near unity (for example $M = \frac{10}{9}$, a value for which the validity of the theory is in question), the flutter calculations become difficult to plot because of the appearance of other branches. In some cases (for instance, $x_0 = 0.6$) the flutter instability appears limited to a definite range of speed. Calculations to include damping were performed to verify the existance of the range. (The appearance of these other branches seems to involve values of $1/k$ for which the quantity M_4 is negative. The condition of negative M_4 is significant for the one-degree-of-freedom instability discussed in the next section.

A plot of the flutter coefficient against Mach number for two values of x_α is shown in figure 17. The significance of the straight lines drawn from the origin has already been discussed. The type of curve A is representative of the effect of forward location of the center of gravity and the type of curve B is representative of rearward locations of the center of gravity. Figure 18 gives a plot of the flutter coefficient against M for various values of the wing-density parameter μ and for a rearward location of the center of gravity. The subsonic values for $M = 0$ and $M = 0.7$ shown on these curves and on some of the other figures have been either taken from reference 7 or calculated in the manner outlined therein. The subsonic and supersonic parts of the curves (figs. 17 and 18) have been arbitrarily joined by a dashed smooth curve in the transonic range. In figure 19 there is given a cross plot of flutter coefficient against frequency ratio ω_h/ω_α, for various values of M, and in figure 20 is given a similar cross plot for three values of the elastic-axis parameter x_0.

An indication of the effect of structural damping in increasing the flutter speed in a few examples may be obtained from the following table, where g_α and g_h are the torsional and flexural damping coefficients, respectively, and where $M = \frac{10}{7}$, $\mu = 7.854$, $a = 0$, and $x_\alpha = 0.2$:

ω_h/ω_α	g_α	g_h	ω/ω_α	$v/b\omega_\alpha$
0	0	0	0.673	2.438
0	.05	0	.648	2.551
0	.10	0	.628	2.669
.707	0	0	.777	1.535
.707	.05	0	.771	1.553
.707	.10	0	.766	1.569
.707	0	.05	.788	1.592
.707	0	.10	.797	1.642
.707	.05	.05	.782	1.623
.707	.10	.10	.784	1.725

STATIC CASES - WING DIVERGENCE AND AILERON REVERSAL

It is of some interest to examine the expressions for the forces and moments in the limit case in which the frequency approaches zero. There follow then for the mean-line wing section the well-known static-case results which may of course be obtained directly without the use of a limiting process, as originally treated by Ackeret. Thus, with the use of the following relation easily verified from equations (20),

$$\lim_{k \to 0} f_\lambda(m, k) = \frac{1}{\lambda + 1}$$

there are obtained from equations (16') to (18') for the lift and moments in the static case,

$$L = -\frac{4\rho b v^2}{\sqrt{M^2 - 1}}\left[\alpha + \left(1 - x_1\right)\beta\right]$$

$$M_\alpha = -\frac{4\rho b^2 v^2}{\sqrt{M^2 - 1}}\left[\left(1 - 2x_0\right)\alpha + \left(1 - x_1\right)\left(1 + x_1 - 2x_0\right)\beta\right]$$

$$M_\beta = -\frac{4\rho b^2 v^2}{\sqrt{M^2 - 1}}\left(1 - x_1\right)^2(\alpha + \beta)$$

These relations for the mean-line wing section are now used to obtain the critical speeds for the static instabilities - wing divergence and wing-aileron reversal (for wing of infinite span). At the wing divergence speed the effective torsional stiffness of the wing vanishes, hence the total moment about the elastic axis is zero. The sum of the structural restoring moment and the aerodynamic twisting moment is

$$\alpha C_\alpha + \frac{4\rho b^2 v^2}{\sqrt{M^2 - 1}}\alpha\left(1 - 2x_0\right)$$

which when equated to zero yields the divergence speed

$$v_D = b\omega_\alpha\left(M^2 - 1\right)^{1/4}\sqrt{\mu r_\alpha^2}\,\frac{1}{\sqrt{2x_0 - 1}}$$

Thus, the divergence speed is real only for positions of the elastic axis behind the aerodynamic center (mid-chord, in the simple theory). This formula obviously should not be used for values of M too near unity.

For comparison it is of interest to note the corresponding result for the divergence speed in the subsonic case, where the aerodynamic center is (approximately) at the quarter-chord point. Thus,

$$v_D = b\omega_\alpha \left(1 - M^2\right)^{1/4} \sqrt{\frac{r_\alpha^2}{\kappa}} \frac{1}{\sqrt{4x_0 - 1}}$$

where $M <$ about 0.7.

The aileron reversal speed is determined by the condition that the change in angle of attack due to wing torsion nullifies the effect of movement of the aileron so as to yield no change in lift (in rolling moment, in the case of finite wing span). There are two equations to be satisfied for this condition; namely,

$$\alpha + \left(1 - x_1\right)\beta = 0$$

(that is, $L = 0$) and

$$\alpha C_\alpha + \frac{4\rho b^2 v^2}{\sqrt{M^2 - 1}} \left[\left(1 - 2x_0\right)\alpha + \left(1 - x_1\right)\left(1 + x_1 - 2x_0\right)\beta\right] = 0$$

The aileron reversal speed, obtained by elimination of α and β, is

$$v_R = b\omega_\alpha \left(M^2 - 1\right)^{1/4} \sqrt{\mu r_\alpha^2} \frac{1}{\sqrt{x_1}}$$

For hinge positions aft of the midchord, the factor $1/\sqrt{x_1}$ in this expression varies from 1.4 to 1.0. The aileron reversal speed is thus relatively unaffected by the position of the hinge. In general v_R may be expected to be lower than v_D.

ONE-DEGREE-OF-FREEDOM OSCILLATORY INSTABILITY

As was pointed out by Possio, the theory indicates the existence of a torsional instability which may arise for a wing having only one degree of freedom. This instability is due to the wing being negatively damped in torsion and is associated with the vanishing (and change in sign) of the torsional damping coefficient M_4 (equation (26)).

Certain considerations for the case of slow oscillations made by Possio (reference 1) and further discussed by Temple and Jahn serve to bring out the main results. Thus from equation (20), for slow oscillations,

$$f_\lambda(M,\ k) \approx \frac{1}{\lambda + 1} - i\frac{2kM^2}{M^2 - 1}\frac{1}{\lambda + 2}$$

and

$$M_4 \approx \frac{1}{\sqrt{M^2 - 1}}\frac{1}{k}\frac{2}{3}\left[4 - 9x_0 + 6x_0^2 - \frac{M^2}{M^2 - 1}\left(2 - 3x_0\right)\right]$$

The condition $M_4\left(M,\ x_0\right) = 0$ is shown plotted in figure 21, where the shaded area is the region in which the instability is possible (negative M_4). The maximum ranges for the parameters x_0 and M in this region are x_0 less than 2/3 and M less than $\sqrt{2.5}$ (and greater than unity).

(It may be appropriate to mention that a similar torsional instability is theoretically indicated even in the subsonic (incompressible) case for positions of the axis of rotation between the leading edge and the quarter-chord point. However, the combination of parameters required for this indicated instability is practically unattainable.)

The torsional instability may be studied more fully in the general case. It is found that the range of instability for the parameters x_0 and M remains essentially as in the simple case (large $1/k$) but more information may be obtained regarding the critical speed and frequency. The moment equation is equivalent to $\overline{A}_{a\alpha} = 0$, or to the two equations

$$\Omega_\alpha X - \mu r_\alpha^2 + M_3\left(M,\ x_0\right) = 0$$

$$M_4\left(M,\ x_0\right) + g_\alpha\Omega_\alpha X = 0$$

where the structural damping coefficient in torsion g_α has been introduced as in reference 6. The critical speed and frequency may be studied as functions of the parameters x_0, M, g_α and the product combination μr_α^2. Results of a few selected calculations are shown plotted in figure 22. Since instabilities are indicated for the range of near-sonic values $(1 < M < 1.58)$ it would seem that a more comprehensive investigation of this problem is very desirable.

It may be remarked that a similar analysis for pure bending exhibits no instability while the case of the aileron alone does exhibit a range where such instability may occur. This range for an aileron hinged at its leading edge is $1 < M \leq \sqrt{2}$.

Langley Memorial Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Field, Va., May 29, 1946

APPENDIX

SYMBOLS

ϕ	disturbance velocity potential
t	time at which disturbance influence is felt
T	time at which disturbance is created
$\tau = t - T$	
p	pressure
p'	pressure difference
ρ	density
γ	adiabatic index (for air, $\gamma \approx 1.4$)
v	velocity of main stream (supersonic)
c	velocity of sound in undisturbed medium
M	Mach number (v/c)
x	coordinate measured in direction of main stream
y	ordinate
x_0	abscissa of axis of rotation of wing section (elastic axis)
x_1	abscissa of aileron hinge
ξ, η	abscissa and ordinate of point of disturbance
b	one-half chord

After equation (12) the quantities x, y, x_0, x_1, and ξ are employed nondimensionally and are referred to the chord 2b as reference length.

$w(x, t)$	vertical velocity at position x on chord and at time t

h	vertical displacement of axis of rotation
α	angular displacement about axis of rotation
β	angular displacement of aileron; measured with respect to α
ω	angular frequency of oscillation
k	reduced frequency $(\omega b/v)$
$\bar{\omega}$	frequency parameter $\left(\frac{2kM^2}{M^2 - 1}\right)$
$I(\xi, x)$	function given in equations (12) and (12')
$J_n(\lambda)$	Bessel function of order n

The following additional symbols, employed in the flutter equations, conform to the notation used in references 4 and 6, in which the half-chord b is the unit reference length.

M	mass of wing per unit span
S_α	static moment of wing-aileron combination per unit span referred to the elastic axis
S_β	static moment of aileron per unit span referred to aileron hinge
I_α	moment of inertia of wing-aileron combination about elastic axis per unit span
I_β	moment of inertia of aileron about its hinge per unit span
a	coordinate of elastic axis measured from midchord $(2x_0 - 1)$
c	coordinate of aileron hinge axis measured from the midchord $(2x_1 - 1)$
x_α	location of center of gravity of wing-aileron system measured from elastic axis S_α/M_b; location of center of gravity in percent total chord measured from leading edge is $100 \frac{1 + a + x_\alpha}{2} = 100\left(x_0 + \frac{x_\alpha}{2}\right)$

x_β reduced location of center of gravity of aileron referred to c $\left(S_\beta/M_b\right)$

r_α radius of gyration of wing-aileron combination referred to a $\left(\sqrt{\frac{I_\alpha}{Mb^2}}\right)$

r_β reduced radius of gyration of aileron referred to c $\left(\sqrt{\frac{I_\beta}{Mb^2}}\right)$

C_α torsional stiffness of wing around a per unit span

C_β torsional stiffness of aileron system around c per unit span

C_h stiffness of wing in deflection

ω_α natural angular frequency of torsional vibrations about elastic axis $\left(\sqrt{\frac{C_\alpha}{I_\alpha}}\right)$; $\left(\omega_\alpha = 2\pi f_\alpha\right.$, where f_α is in cycles per second$\left.\right)$

ω_β natural angular frequency of torsional vibrations of aileron around c $\left(\sqrt{\frac{C_\beta}{I_\beta}}\right)$

ω_h natural angular frequency of wing in deflection $\left(\sqrt{\frac{C_h}{M}}\right)$

$\mu = \frac{\pi}{4}\frac{1}{\kappa}$ wing density parameter, where $\kappa = \frac{\pi\rho b^2}{M}$ is the ratio of a mass of cylinder of air of a diameter equal to the chord of the wing to the mass of the wing, both taken for equal length along the span; this ratio may be expressed as $\kappa = 0.24\left(b^2/W\right)\left(\rho/\rho_o\right)$ where W is the weight in pounds per foot span, b is in feet and ρ/ρ_o is the ratio of air density at altitude to that for normal standard air $\left(\mu = \frac{M}{4\rho b^2} = \frac{\pi}{4}\frac{1}{\kappa}\right)$

g_α, g_β, g_h structural damping coefficients (see reference 6)

$L_1, L_2, L_3, L_4, M_1, M_2, M_3, M_4$ quantities defined in table II and by equations (26) and (28)

$v/b\omega_\alpha$ flutter coefficient; that is, flutter speed divided by reference speed $b\omega_\alpha$

$$\Omega_\alpha X = \mu r_\alpha^2\left(\frac{\omega_\alpha}{\omega}\right)^2$$

$$\Omega_\beta X = \mu r_\beta^2\left(\frac{\omega_\beta}{\omega}\right)^2$$

$$\Omega_h X = \mu\left(\frac{\omega_h}{\omega}\right)^2$$

where ω is the angular (flutter) frequency and

$$X = \mu r_\alpha^2\left(\frac{\omega_\alpha}{\omega}\right)^2$$

for case of bending-torsion (Note that in the incompressible case (references 4 and 6) μ is replaced by $1/\kappa$.)

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TABLE I.- VALUES OF $f_0(M, \overline{\omega}) = (f_0)_R + i(f_0)_I$

$$(f_0)_R = \frac{1}{\overline{\omega}} \int_0^{\overline{\omega}} J_0\left(\frac{u}{M}\right) \cos u \, du$$

$$(f_0)_I = -\frac{1}{\overline{\omega}} \int_0^{\overline{\omega}} J_0\left(\frac{u}{M}\right) \sin u \, du$$

$\overline{\omega}$	$\frac{1}{k}$	$(f_0)_R$	$(f_0)_I$	$\overline{\omega}$	$\frac{1}{k}$	$(f_0)_R$	$(f_0)_I$
$M = \frac{10}{9}$				$M = \frac{10}{7}$			
20.00	0.526	0.02107622	-0.14998785	20.00	0.196	0.01041793	-0.05473581
10.00	1.053	.10786366	-.21774161	10.00	.392	-.02790057	-.18976570
8.00	1.316	.15423397	-.24646721	5.00	.784	.13530140	-.33798972
6.00	1.754	.17593123	-.27930193	3.90	1.006	.15895143	-.35747165
5.00	2.105	.23364109	-.30623887	3.10	1.265	.19688319	-.44343786
4.20	2.506	.26927908	-.29526037	2.40	1.634	.32317741	-.53918653
3.20	3.289	.27020078	-.32995569	2.00	1.961	.44414008	-.56786346
2.50	4.211	.32453905	-.42367574	1.60	2.451	.59012790	-.55477283
2.10	5.013	.40860905	-.47752247	1.18	3.323	.749343	-.482889
1.68	6.266	.539882	-.502746	.94	4.172	.832504	-.413781
1.40	7.519	.64370304	-.48895719	.78	5.028	.881388	-.357217
1.10	9.569	.75786234	-.44031451	.62	6.325	.923323	-.293258
.52	20.243	.938879	-.247350	.52	7.541	.945429	-.250038
.26	40.486	.984312	-.128388	.40	9.804	.967340	-.195428
.10	105.263	.997660	-.049910	.20	19.608	.991735	-.099425
				.10	39.216	.997930	-.049930
				.04	98.039	.999675	-.020000
$M = \frac{5}{4}$				$M = \frac{5}{3}$			
20.00	0.278	-0.02589034	-0.08629977	20.00	0.156	0.00827247	-0.07001922
10.00	.556	.02529654	-.22399799	10.00	.312	-.03440806	-.13439143
5.00	1.111	.19004335	-.33199853	5.00	.625	.07303312	-.32465775
4.40	1.263	.21539228	-.32383270	3.10	1.008	.15700194	-.49133028
3.30	1.684	.22643569	-.36922184	2.50	1.250	.28500372	-.57452410
2.80	1.984	.26380752	-.43392085	1.90	1.645	.48400601	-.60117841
2.20	2.525	.38184436	-.51318210	1.60	1.953	.59909796	-.57737682
1.66	3.347	.557683	-.532604	1.30	2.404	.71425408	-.52373582
1.34	4.146	.678777	-.500827	.94	3.324	.839483	-.419702
1.10	5.051	.76855505	-.45177830	.76	4.112	.892157	-.352880
.88	6.313	.844260	-.387852	.62	5.040	.926973	-.295089
.74	7.508	.886947	-.338397	.50	6.250	.951928	-.242110
.56	9.921	.933477	-.266032	.42	7.440	.965855	-.205300
.28	19.841	.982907	-.138218	.32	9.766	.980047	-.157913
.14	39.683	.995700	-.069779	.16	19.531	.994975	-.079738
.06	92.593	.999200	-.029983	.08	39.062	.998738	-.039962
				.04	78.125	.999675	-.020000

TABLE I.- VALUES OF $f_0(M, \overline{\omega}) = (f_0)_R + i(f_0)_I$ - Concluded

$\overline{\omega}$	$\frac{1}{k}$	$(f_0)_R$	$(f_0)_I$	$\overline{\omega}$	$\frac{1}{k}$	$(f_0)_R$	$(f_0)_I$
$M = 2$				$M = 5$			
20.00	0.133	-0.01548798	-0.06337016	20.00	0.104	-0.1854996	-0.06011798
10.00	.267	-.00697070	-.08613197	10.00	.208	-.00207595	-.12707440
5.00	.533	.00782279	-.29541589	5.00	.417	-.15446620	-.17734818
2.70	.988	.21306136	-.59458735	4.20	.496	-.16645177	-.35826002
2.10	1.270	.41159770	-.63550404	2.10	.992	.41122567	-.70315051
1.60	1.667	.60680594	-.59698731	1.70	1.225	.58072358	-.65532307
1.30	2.051	.72237641	-.53566558	1.20	1.736	.77380710	-.52771202
1.10	2.424	.79366285	-.47882893	1.00	2.083	.83908297	-.45746891
.80	3.333	.885888	-.371721	.84	2.480	.884607	-.394511
.64	4.167	.925636	-.305330	.62	3.360	.936018	-.299626
.54	4.938	.946567	-.261128	.50	4.167	.958080	-.244532
.42	6.349	.967381	-.205798	.42	4.960	.970300	-.206750
.36	7.407	.975947	-.177347	.34	6.127	.980474	-.168271
.26	10.256	.987392	-.128996	.28	7.440	.986729	-.139032
.14	19.048	.996329	-.069843	.20	10.417	.993215	-.099645
.06	44.444	.999333	-.029983	.10	20.833	.998300	-.049960
.02	133.333	.99950	-.010000	.06	34.722	.999383	-.029983
				.02	104.167	.999950	-.010000
$M = \frac{5}{2}$				$M = \frac{10}{3}$			
20.00	0.119	0.00671539	-0.04537548	20.00	0.110	0.00960890	-0.05304109
10.00	.238	.02216359	-.06784796	10.00	.220	.02509080	-.08576219
5.00	.476	-.05520172	-.25614005	5.00	.440	-.11086610	-.21422339
4.80	.496	-.05938671	-.27691360	4.40	.500	-.12438944	-.31880518
2.40	.992	.29706773	-.65923330	2.20	.999	.37034955	-.68956543
1.90	1.253	.49182591	-.65290610	1.80	1.221	.53617331	-.65968944
1.40	1.701	.69153899	-.57126881	1.30	1.691	.73436415	-.55334513
1.20	1.984	.76519653	-.51684512	1.10	1.998	.80420000	-.49023486
.96	2.480	.844299	-.436326	.88	2.498	.871331	-.408825
.72	3.307	.909960	-.341204	.66	3.330	.926118	-.316656
.58	4.105	.940834	-.280086	.52	4.227	.953675	-.253427
.48	4.960	.959181	-.234350	.44	4.995	.966677	-.216005
.38	6.266	.974266	-.187184	.36	6.105	.977606	-.177806
.32	7.440	.981697	-.158316	.30	7.326	.984410	-.148727
.24	9.921	.989675	-.119288	.22	9.990	.991595	-.109495
.12	19.841	.997408	-.059908	.10	21.978	.998260	-.049950
.06	39.683	.999350	-.029983	.06	36.630	.999367	-.029983
.02	119.048	.999950	-.010000	.02	109.890	.999950	-.010000

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TABLE II.- VALUES OF FUNCTIONS USED IN THE FLUTTER CALCULATIONS

The expressions employed in the calculation of this table are:

$$L_1 = \frac{1}{\sqrt{M^2-1}}\left\{-2\left(f_0\right)_R + \frac{1}{k}\left[J_0\left(\frac{\bar{\omega}}{M}\right)\sin\bar{\omega} - \frac{1}{M}J_1\left(\frac{\bar{\omega}}{M}\right)\cos\bar{\omega}\right]\right\} \qquad M_1' = L_1 - A_1$$

$$L_2 = \frac{1}{\sqrt{M^2-1}}\left\{-2\left(f_0\right)_I + \frac{1}{k}\left[J_0\left(\frac{\bar{\omega}}{M}\right)\cos\bar{\omega} + \frac{1}{M}J_1\left(\frac{\bar{\omega}}{M}\right)\sin\bar{\omega}\right]\right\} \qquad M_2' = L_2 - A_2$$

$$L_3' = L_1 + \frac{1}{k}L_2 + A_1 \qquad M_3' = \frac{4}{3}\left(L_1 - B_1\right) + \frac{1}{k}\left(L_2 + A_2\right)$$

$$L_4' = L_2 - \frac{1}{k}L_1 + A_2 \qquad M_4' = \frac{4}{3}\left(L_2 - B_2\right) - \frac{1}{k}\left(L_1 + A_1\right)$$

where

$$A_1 = \frac{1}{\sqrt{M^2-1}}\frac{1}{M}\frac{2}{2k^2}\left[\frac{1}{M}\left(f_0\right)_R - \frac{1}{M}J_0\left(\frac{\bar{\omega}}{M}\right)\cos\bar{\omega} - J_1\left(\frac{\bar{\omega}}{M}\right)\sin\bar{\omega}\right], \qquad B_1 = \frac{1}{\sqrt{M^2-1}}\frac{1}{M}\frac{1}{2k^2}\left[-\frac{2}{\bar{\omega}}J_1\left(\frac{\bar{\omega}}{M}\right)\cos\bar{\omega} + \frac{1}{M}J_0\left(\frac{\bar{\omega}}{M}\right)\cos\bar{\omega} + J_1\left(\frac{\bar{\omega}}{M}\right)\sin\bar{\omega}\right]$$

$$A_2 = \frac{1}{\sqrt{M^2-1}}\frac{1}{M}\frac{1}{2k^2}\left[\frac{1}{M}\left(f_0\right)_I + \frac{1}{M}J_0\left(\frac{\bar{\omega}}{M}\right)\sin\bar{\omega} - J_1\left(\frac{\bar{\omega}}{M}\right)\cos\bar{\omega}\right] \qquad B_2 = \frac{1}{\sqrt{M^2-1}}\frac{1}{M}\frac{1}{2k^2}\left[\frac{2}{\bar{\omega}}J_1\left(\frac{\bar{\omega}}{M}\right)\sin\bar{\omega} - \frac{1}{M}J_0\left(\frac{\bar{\omega}}{M}\right)\sin\bar{\omega} + J_1\left(\frac{\bar{\omega}}{M}\right)\cos\bar{\omega}\right]$$

$\frac{1}{k}$	L_1	L_2	L_3'	L_4'	M_1'	M_2'	M_3'	M_4'	$M_1 + L_3'$	$M_2' + L_4'$	D_R	D_I
$M = \frac{10}{9}$												
0.526	-0.02525	0.44559	0.25959	0.44106	-0.07557	0.46341	0.24942	0.60938	0.18402	0.90447	-0.05382	0.00879
1.053	.06401	.80255	1.07622	.79116	-.10309	.74716	1.02212	1.20708	.97313	1.53832	-.20161	.17533
1.316	.17553	1.03257	1.73315	.88014	-.02346	.95403	1.65937	1.43563	1.70969	1.83417	-.31078	.33258
1.754	.39623	1.32462	2.99583	.88853	.11753	1.06558	2.65307	1.76531	3.11636	1.95411	-.69014	.91306
2.105	.62771	1.73618	4.57331	.68882	.33722	1.46206	4.15796	1.77406	4.91053	2.15087	-1.00521	1.41382
2.506	.74997	2.15463	6.63529	.62547	.26471	1.80417	6.00140	2.37986	6.90000	2.42964	-1.25482	2.57886
3.289	1.30403	2.69560	10.88273	-.62644	.59242	1.72808	8.27851	2.42162	11.47515	1.10164	-3.26198	7.03830
4.211	2.52780	3.74107	18.66776	-5.21239	2.13970	2.05114	12.62101	-2.42793	20.80746	-3.16125	-9.64830	13.94152
5.013	3.61621	5.14361	29.31903	-10.94724	3.69590	3.10809	20.55344	-9.65451	33.01493	-7.83915	-18.40062	20.13986
6.266	4.97649	7.84048	53.34317	-21.13670	5.73565	5.65664	41.39481	-23.23245	59.07882	-15.50006	-36.94330	29.49559
7.519	5.92672	10.77296	85.65512	-31.63176	7.19805	8.61585	71.57563	-37.52002	92.85315	-23.01591	-60.67223	38.40544
9.569	6.89867	15.63756	154.72304	-48.43295	8.71601	13.69210	138.52116	-60.51579	163.43905	-34.74085	-109.78529	52.31323
20.245	8.35744	39.58910	807.07986	-128.09476	10.99388	38.49893	787.81916	-168.60818	818.07374	-89.59585	-561.00925	119.83831
40.486	8.68383	82.45879	3344.25646	-268.54746	11.54373	81.89315	3324.26400	-357.96892	3355.78019	-186.65431	-2286.34563	244.04260
105.263	8.78460	216.90161	22837.60050	-707.59822	11.71669	216.70691	22814.80624	-942.10469	22849.31720	-490.89131	-16159.53823	-482.93475
$M = \frac{5}{4}$												
0.278	-0.00103	0.22815	0.06045	0.21882	0.00087	0.23777	0.05814	0.29553	0.06132	0.45659	-0.01551	-0.00160
.556	-.01996	.41500	.25390	.41670	-.06327	.42439	.24339	.57278	.19063	.84109	-.04965	.00819
1.111	.07962	.79349	1.07903	.74308	-.03812	.75543	1.04705	1.10037	1.04091	1.49851	-.18729	.13165
1.263	.09225	.89629	1.39479	.84102	-.07860	.83508	1.34400	1.30141	1.31619	1.67610	-.23051	.22601
1.684	.25179	1.11026	2.36647	.94772	.00623	.84890	2.05296	1.67536	2.37270	1.79662	-.55339	.68636
1.984	.47142	1.32708	3.30046	.80052	.27547	.91828	2.70703	1.59376	3.57593	1.71880	-1.01298	1.09251
2.525	.89075	1.91297	5.74578	.22823	.86646	1.34853	4.68307	.91962	6.61224	1.57656	-2.25852	1.83276
3.347	1.37142	3.05697	11.38361	-.91551	1.59007	2.43968	9.82891	-.61189	12.97368	1.52417	-4.98420	2.89088
4.146	1.66481	4.25359	18.92040	-2.06073	2.04435	3.66571	17.07927	-2.19177	20.96475	1.60498	-8.47733	3.85549
5.051	1.87067	5.61041	29.70974	-3.30446	2.36705	5.07747	27.67224	-3.91426	32.07677	1.77301	-13.37591	4.90177
6.313	2.03842	7.46934	48.59984	-4.94077	2.63194	7.01066	46.40466	-6.17197	51.23178	2.06989	-21.85719	6.31797
7.508	2.13102	9.19016	70.47841	-6.40790	2.77884	8.78960	68.19697	-8.18404	73.25725	2.38170	-31.62936	7.63021
9.921	2.23050	12.58897	126.41442	-9.22392	2.93717	12.27385	124.04105	-12.01794	129.35159	3.04993	-56.54665	10.24338
19.841	2.33469	26.12636	519.94592	-20.03286	3.10371	25.96238	517.48047	-26.56078	523.04965	5.92952	-231.76910	21.01250
39.683	2.36140	52.74454	2094.61529	-40.88085	3.14472	52.66366	2092.04724	-54.48923	2097.76001	11.78281	-925.74406	33.84983
92.593	2.36874	123.38572	11426.15260	-95.90886	3.18859	123.35267	11423.45420	-125.00012	11429.34119	27.44581	-5781.50717	54.41156
$M = 2$												
0.133	-0.00008	0.06698	0.00885	0.06657	-0.00009	0.06739	0.00881	0.08885	0.00876	0.13396	-0.00146	-0.00001
.267	.00177	.13612	.03543	.13288	.00439	.13889	.03521	.17813	.03982	.27177	-.00588	-.00040
.533	-.01646	.26350	.14409	.26138	-.03647	.27440	.14254	.35343	.10762	.53578	-.01849	.00168
.988	.03592	.44538	.49654	.45368	.01517	.40159	.47173	.62041	.51172	.85528	-.08471	.02609
1.270	.07918	.60315	.84503	.55683	.07924	.54892	.80838	.75833	.92426	1.10575	-.15466	.03964
1.667	.11850	.84295	1.50219	.69968	.13972	.78872	1.45559	.94730	1.64190	1.48840	-.28407	.05667
2.051	.14082	1.07856	2.31942	.83947	.17466	1.02878	2.26740	1.13198	2.49408	1.86825	-.44308	.07213
2.424	.15433	1.39577	3.27821	.97664	.19595	1.26078	3.22296	1.31341	3.47416	2.23742	-.62866	.08668
3.333	.17153	1.85225	6.29401	1.31584	.22322	1.81690	6.23471	1.76306	6.51723	3.13275	-1.21042	.12113
4.167	.17886	2.34649	9.89587	1.63041	.23489	2.31733	9.83886	2.18092	10.13476	3.94774	-1.90491	.15263
4.938	.18270	2.80063	13.95465	1.92342	.24102	2.77564	13.89275	2.57055	14.19567	4.69905	-2.68556	.18150
6.349	.18651	3.62600	23.14814	2.46157	.24710	3.60626	23.08537	3.28681	23.39524	6.06783	-4.45516	.23411
7.407	.18807	4.24247	31.55225	2.86638	.24960	4.22544	31.48909	3.82539	31.80185	7.09182	-6.07291	.27360
10.256	.19016	5.89669	60.60623	3.95877	.25294	5.88429	60.54270	5.28131	60.85917	9.84306	-11.66499	.37971
19.048	.19178	10.98370	209.34139	7.33738	.25559	10.97698	209.27740	9.78561	209.59698	18.31436	-49.30930	.70514
44.444	.19232	25.65424	1140.31777	17.11024	.25550	25.65060	1140.28154	22.76366	1140.57327	42.76084	-217.14303	3.23223
133.333	.19243	76.97811	10263.90867	51.32263	.22446	76.97626	10263.73640	64.15592	10264.13312	128.29889	-1316.71251	6.52232
$M = \frac{5}{2}$												
0.119	0.00029	0.047	0.00567	0.04760	0.00059	0.04776	0.00567	0.06351	0.00626	0.09536	-0.00076	-0.00001
.238	.00080	.095	.02271	.09517	.00160	.09532	.02273	.12728	.02431	.19049	-.00308	-.00005
.476	-.01004	.190	.09083	.18862	-.02004	.19777	.08981	.25367	.07079	.38639	-.01018	.00041
.496	-.01083	.1967	.09847	.19654	-.02254	.20253	.09718	.26426	.07593	.39887	-.01106	.00074
.992	.02406	.37551	.40186	.37419	.01879	.35297	.39036	.50433	.42065	.72716	-.05546	.00984
1.253	.04114	.49084	.65264	.46579	.04474	.46653	.63795	.62350	.69738	.93012	-.09271	.01369
1.701	.05800	.69385	1.22505	.61784	.07098	.67123	1.20742	.82810	1.29603	1.28907	-.17679	.01965
1.984	.06409	.82238	1.67931	.71598	.08056	.80161	1.66066	.95850	1.75987	1.51759	-.24317	.02329
2.480	.07056	1.04583	2.64413	.88859	.09081	1.02807	2.62441	1.18800	2.73494	1.91666	-.38385	.02947
3.307	.07589	1.41473	4.73083	1.17780	.09928	1.40071	4.71024	1.57289	4.85011	2.57851	-.68768	.03963
4.105	.07838	1.76825	7.31232	1.45804	.10525	1.75669	7.29132	1.94610	7.41557	3.21473	-1.06337	.04939
4.960	.07786	2.14533	10.69564	1.75892	.10561	2.13564	10.67440	2.54692	10.80125	3.89456	-1.55561	.05974
6.266	.08107	2.71898	17.09078	2.21879	.10754	2.71122	17.06935	2.95974	17.19832	4.93001	-2.48598	.07569
7.440	.08166	3.23412	24.11819	2.63306	.10850	3.22755	24.09665	3.51190	24.22669	5.86061	-3.50866	.08989
9.921	.08230	4.31980	42.91023	3.50826	.10951	4.31485	42.88858	4.67848	43.01974	7.82311	-6.24185	.11973
19.841	.08293	8.55446	171.77072	7.01164	.11055	8.65193	171.74976	9.34996	171.88127	15.66357	-25.00068	.25344
39.683	.08308	17.31638	687.21329	14.02108	.11088	17.31481	687.20445	18.69929	687.32417	31.33589	-100.13694	.92473
119.048	.08311	51.95581	6185.28165	42.06234	.09993	51.95562	6185.22715	54.78641	6185.38158	94.01796	-765.13832	-1.30658

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TABLE II.- VALUES OF FUNCTIONS USED IN FLUTTER CALCULATIONS - Concluded

$\frac{1}{k}$	L_1	L_2	L_3'	L_4'	M_1'	M_2'	M_3'	M_4'	$M_1' + L_3'$	$M_2' + L_4'$	D_R	D_I
$M = \frac{10}{7}$												
0.196	0.00227	0.13710	0.02700	0.13688	0.00443	0.13688	0.02705	0.18353	0.03145	0.27376	-0.00648	-0.00018
.392	-.00911	.27591	.10722	.26624	-.01725	.28916	.10315	.35992	.08997	.55540	-.02141	-.00124
.784	-.00598	.50880	.45680	.50919	-.06970	.51311	.45040	.70878	.38710	1.02230	-.07021	.02603
1.006	.01036	.60695	.73498	.64457	-.10395	.55892	.69894	.93421	.63103	1.20349	-.12511	.09011
1.265	.10629	.73206	1.14404	.73643	-.00539	.59322	1.02716	1.10167	1.13865	1.32965	-.25428	.19434
1.634	.28611	1.00994	1.98049	.76349	.24197	.78890	1.74282	1.16631	2.22246	1.55239	-.55617	.34669
1.961	.42538	1.32284	2.99507	.74175	.44550	1.07377	2.67381	1.14150	3.44057	1.81552	-.91582	.47385
2.451	.57188	1.83902	4.97860	.68975	.67257	1.58663	4.57135	1.06421	5.65117	2.27638	-1.59692	.65230
3.323	.72169	2.78331	9.78696	.60896	.90636	2.55920	9.29616	.93408	10.69332	3.16816	-3.20295	.94949
4.172	.79648	3.68899	15.95861	.55943	1.02440	3.49574	15.42703	.84909	16.98301	4.05517	-5.23734	1.22621
5.028	.83956	4.58539	23.64012	.53171	1.09272	4.41806	23.08528	.79732	24.73284	4.94977	-7.75745	1.49993
6.325	.87604	5.92041	38.04859	.51702	1.15075	5.78275	37.47423	.76135	39.19934	6.29977	-12.47320	1.90933
7.541	.89511	7.15470	54.56608	.52171	1.18116	7.03720	53.98156	.75680	55.74724	7.55891	-17.87514	2.29065
9.804	.91592	9.42482	93.01679	.55672	1.21121	9.33290	92.42232	.79027	94.22800	9.88962	-30.44860	2.99527
19.608	.93472	19.12578	375.64051	.84447	1.24431	19.07920	375.03141	1.14846	376.88482	19.92367	-122.71726	5.87053
39.216	.93997	38.39203	1506.19800	1.55268	1.25171	38.36973	1505.55164	2.04124	1507.44971	39.92241	-488.94116	8.74834
98.039	.94143	96.07882	9420.13594	3.77957	1.23887	96.08070	9418.39694	3.45911	9421.37481	99.86027	-2772.76586	-186.21681
$M = \frac{5}{3}$												
0.156	-0.00090	0.09297	0.01471	0.09352	-0.00198	0.09255	0.01474	0.12493	0.01273	0.18607	-0.00294	0.00008
.312	-.00024	.19313	.05802	.18525	.00185	.20108	.05672	.24911	.05987	.38633	-.01098	-.00111
.625	-.01971	.36096	.24221	.36086	-.05602	.37339	.23936	.49241	.18619	.73425	-.03415	.00647
1.008	.03778	.51751	.61729	.54668	-.02005	.45027	.57390	.76842	.59724	.99695	-.11745	.05904
1.250	.11109	.66519	.97181	.62689	.08187	.56462	.89651	.88199	1.05368	1.19151	-.21271	.09430
1.645	.20716	.96017	1.76808	.73691	.22547	.84272	1.65534	1.02910	1.99355	1.57963	-.42283	.14645
1.953	.25811	1.20754	2.57072	.81960	.30398	1.09136	2.43919	1.13661	2.87470	1.91096	-.62989	.18407
2.404	.30697	1.57362	4.01643	.94318	.38027	1.46616	3.86735	1.29656	4.39670	2.40934	-.99761	.23635
3.324	.35828	2.31278	7.94412	1.20910	.46121	2.22538	7.77701	1.64279	8.40533	3.43448	-1.98627	.33876
4.112	.37939	2.93308	12.32441	1.44693	.49472	2.85924	12.15000	1.95482	12.81913	4.30617	-3.08407	.42429
5.040	.39321	3.65468	18.69044	1.73469	.51675	3.59275	18.51127	2.33422	19.20717	5.32744	-4.67761	.52416
6.250	.40305	4.58479	28.92859	2.11660	.53243	4.53388	28.74605	2.83956	29.46102	6.65048	-7.23873	.65339
7.440	.40852	5.49340	41.14941	2.49701	.54116	5.45019	40.96502	3.34409	41.69057	7.94720	-10.29465	.77998
9.766	.41408	7.25745	71.15166	3.24691	.55003	7.22420	70.96515	4.34048	71.70169	10.47111	-17.79479	1.02361
19.531	.41992	14.61478	285.72525	6.43000	.55947	14.59798	285.53667	8.58100	286.28472	21.02798	-71.49656	2.05007
39.062	.42139	29.28000	1144.03031	12.82878	.56254	29.27048	1143.88498	17.13850	1144.59285	42.09926	-287.85185	6.64120
78.125	.42178	58.58531	4577.25033	25.63499	.57077	58.58433	4576.81368	34.84586	4577.82110	84.21932	-1221.79608	-21.02996
$M = \frac{10}{3}$												
0.110	-0.00012	0.03287	0.00363	0.03296	-0.00026	0.03279	0.00363	0.04396	0.00337	0.06575	-0.00036	0
.220	-.00010	.06520	.01451	.06591	-.00039	.06451	.01453	.08800	.01412	.13042	-.00148	.00003
.440	-.00468	.13262	.05779	.13131	-.00885	.13599	.05732	.17594	.04894	.26730	-.00523	.00008
.500	-.00527	.14749	.07452	.14885	-.01138	.14876	.07360	.19960	.06314	.29761	-.00684	.00041
.999	.01148	.29222	.30418	.28988	.01070	.28310	.29991	.38813	.31488	.57298	-.03117	.00288
1.221	.01675	.36313	.45807	.35199	.01882	.35381	.45309	.47080	.47689	.70580	-.04746	.00372
1.691	.02295	.51431	.88688	.48373	.02851	.50608	.88115	.64618	.91539	.98981	-.09259	.00538
1.998	.02511	.61299	1.24305	.57022	.03192	.60561	1.23704	.76134	1.27497	1.17583	-.12998	.00641
2.498	.02717	.77266	1.94886	.71103	.03520	.76642	1.94260	.94891	1.98406	1.47745	-.20406	.00808
3.330	.02885	1.03736	3.47426	.94617	.03787	1.03249	3.46780	1.26222	3.51213	1.97866	-.36399	.01080
4.227	.02969	1.32129	5.60466	1.19972	.03921	1.31738	5.59811	1.60015	5.64387	2.51710	-.58733	.01373
4.995	.03009	1.56413	7.83317	1.41719	.03985	1.56079	7.82656	1.89003	7.87302	2.97798	-.82097	.01622
6.105	.03042	1.91440	11.70789	1.73144	.04038	1.91164	11.70125	2.30895	11.74827	3.64308	-1.22718	.01992
7.326	.03063	2.29929	16.86519	2.07722	.04071	2.29699	16.85853	2.76995	16.90590	4.37421	-1.76772	.02376
9.990	.03085	3.13830	31.37233	2.83185	.04106	3.13660	31.36567	3.77602	31.41339	5.96845	-3.28842	.03265
21.978	.03105	6.91021	151.89352	6.22861	.04138	6.90942	151.88720	8.30479	151.93490	13.13803	-15.92102	.07645
36.630	.03109	11.51868	421.95016	10.38054	.04156	11.51808	421.94825	13.84493	421.99172	21.89862	-44.32931	.23019
109.890	.03109	34.55853	3797.66470	31.14219	.03809	34.55844	3797.64832	41.15315	3797.70279	65.70063	-372.55103	-.13103
$M = 5$												
0.104	-0.00002	0.02084	0.00217	0.02083	-0.00004	0.02085	0.00216	0.02778	0.00213	0.04168	-0.00014	0
.208	-.00022	.04122	.00868	.04165	-.00053	.04084	.00868	.05556	.00815	.08249	-.00059	.00001
.417	-.00147	.08369	.03463	.08321	-.00270	.08477	.03448	.11119	.03193	.16798	-.00221	.00001
.496	-.00159	.09822	.04905	.09886	-.00351	.09836	.04872	.13215	.04554	.19722	-.00316	.00010
.992	.00350	.19647	.19843	.19557	.00347	.19389	.19732	.26113	.20190	.38946	-.01338	.00053
1.225	.00495	.24454	.30387	.24102	.00572	.24199	.30260	.32169	.30959	.48301	-.02058	.00068
1.736	.00659	.34988	.61232	.34059	.00830	.34772	.61089	.45437	.62062	.68831	-.04160	.00099
2.083	.00715	.42136	.88295	.40836	.00917	.41947	.88148	.54470	.89212	.82783	-.06001	.00120
2.480	.00753	.52090	1.25255	.48586	.00979	.50125	1.25104	.64800	1.26234	.98711	-.08518	.00143
3.360	.00797	.68335	2.30166	.65785	.01048	.68209	2.30011	.87727	2.31214	1.33994	-.15656	.00194
4.167	.00815	.84844	3.54068	.81550	.01078	.84740	3.53912	1.08745	3.55146	1.66290	-.24091	.00243
4.960	.00286	1.01076	5.01927	.97069	.01093	1.00989	5.01770	1.29435	5.03020	1.98058	-.34140	.00286
6.127	.00834	1.24933	7.66080	1.19894	.01108	1.24862	7.65922	1.59866	7.67188	2.44756	-.52124	.00351
7.440	.00839	1.51760	11.29727	1.45574	.01116	1.51701	11.29569	1.94105	11.30843	2.97275	-.76867	.00431
10.417	.00845	2.12544	22.14570	2.05787	.01125	2.12502	22.14412	2.71720	22.15695	4.16289	-1.50675	.00598
20.833	.00849	4.25216	88.59237	4.07543	.01132	4.25196	88.59065	5.43398	88.60369	8.32739	-6.02832	.01041
34.722	.00850	7.08739	246.09543	6.79234	.01136	7.08722	246.09514	9.05733	246.10679	13.87956	-16.75786	.03961
104.167	.00850	21.26285	2214.88627	20.37767	.01054	21.26282	2214.88243	27.08693	2214.89681	41.64049	-147.17700	.00026

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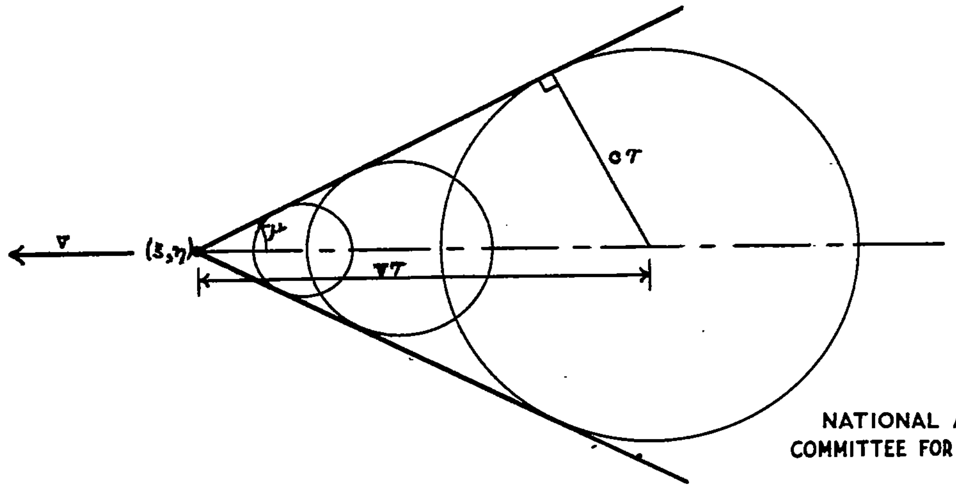

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Figure 1.- Mach angle μ. The disturbance at point (ξ,η) moving forward with supersonic velocity v influences the angular region having half vertex angle $\mu = \sin^{-1}\frac{c}{v}$.

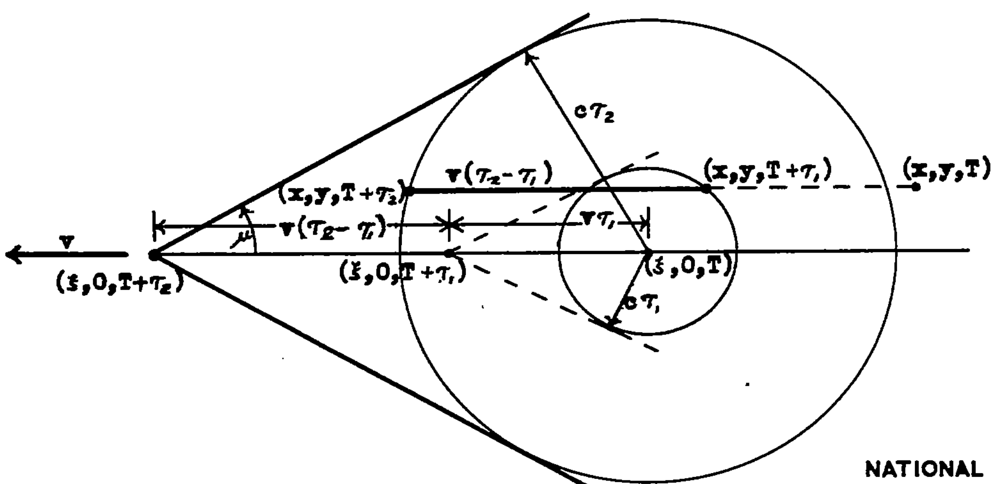

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Figure 2.- Influence of impulse created at point $(\xi,0)$ at time $t = T$ on a point (x,y) fixed relative to $(\xi,0)$ and moving with supersonic velocity v. (Observe that the disturbance influences the point (x,y) only during the time interval $\tau_2 - \tau_1$.)

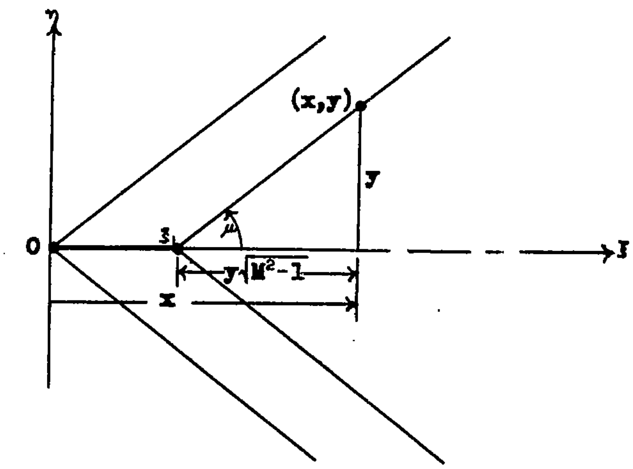

Figure 3.- Sketch showing that only disturbances created forward of the Mach angle region with vertex at ξ_1, can affect (x,y).

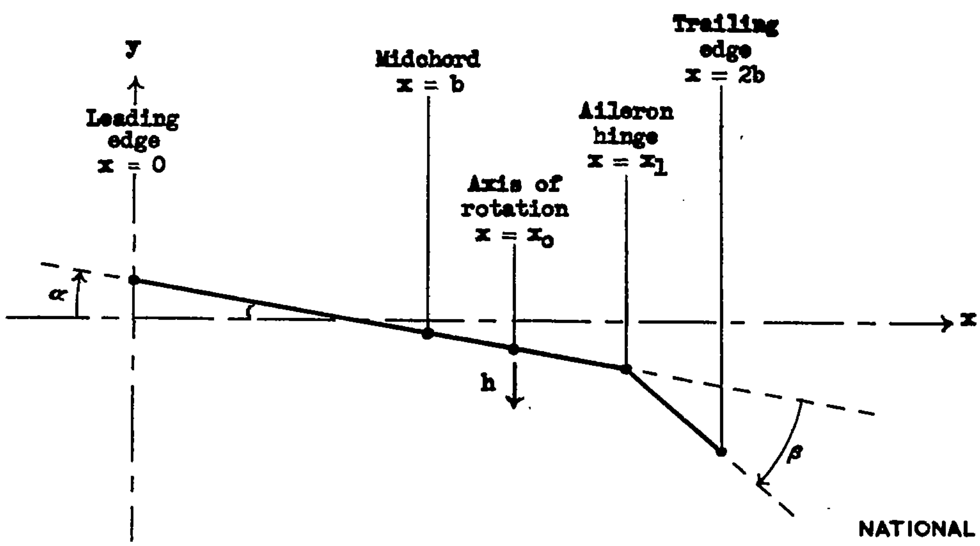

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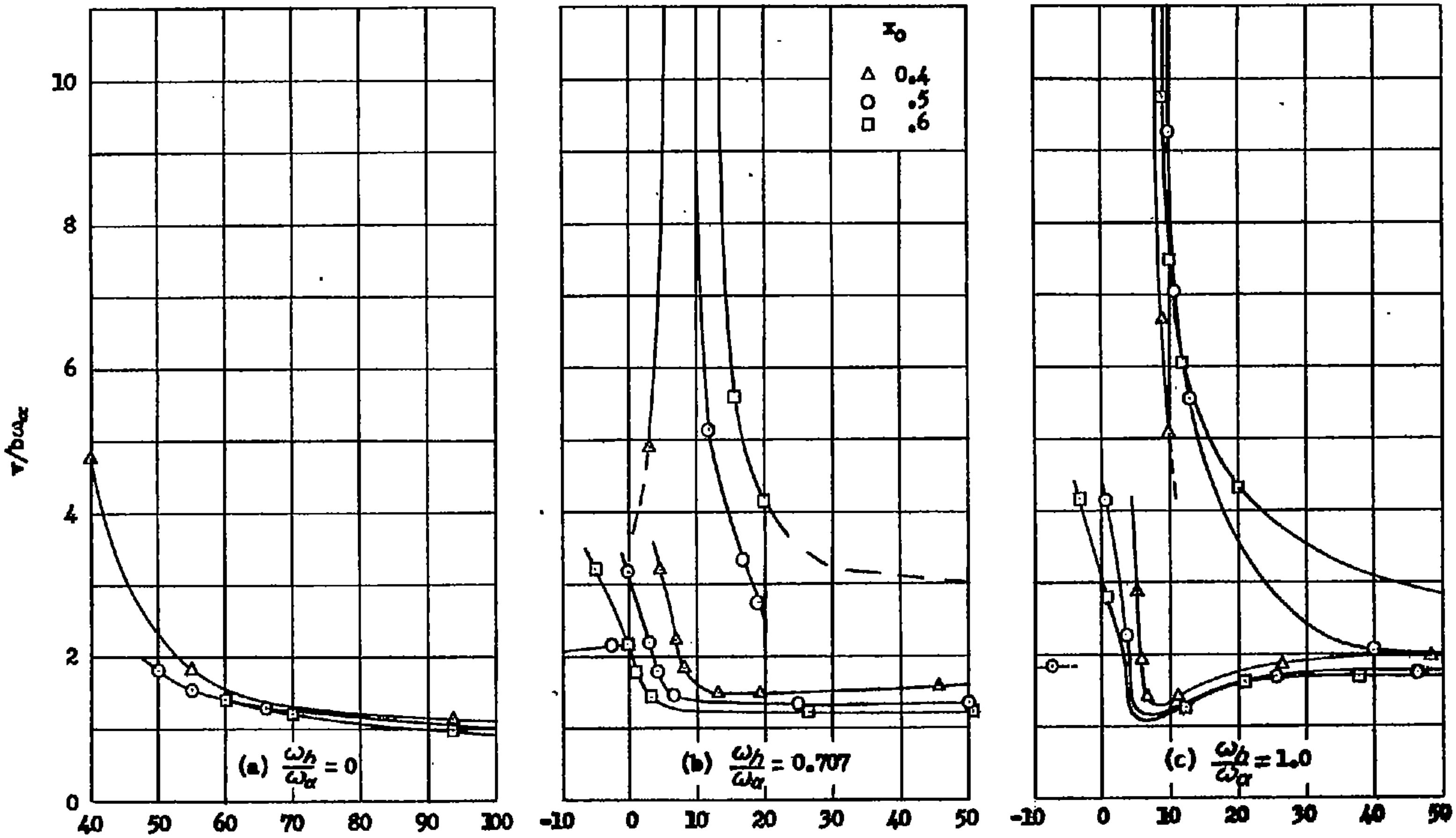
Figure 4.- Sketch illustrating the three degrees of freedom h, α, and β of the oscillating airfoil.



Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

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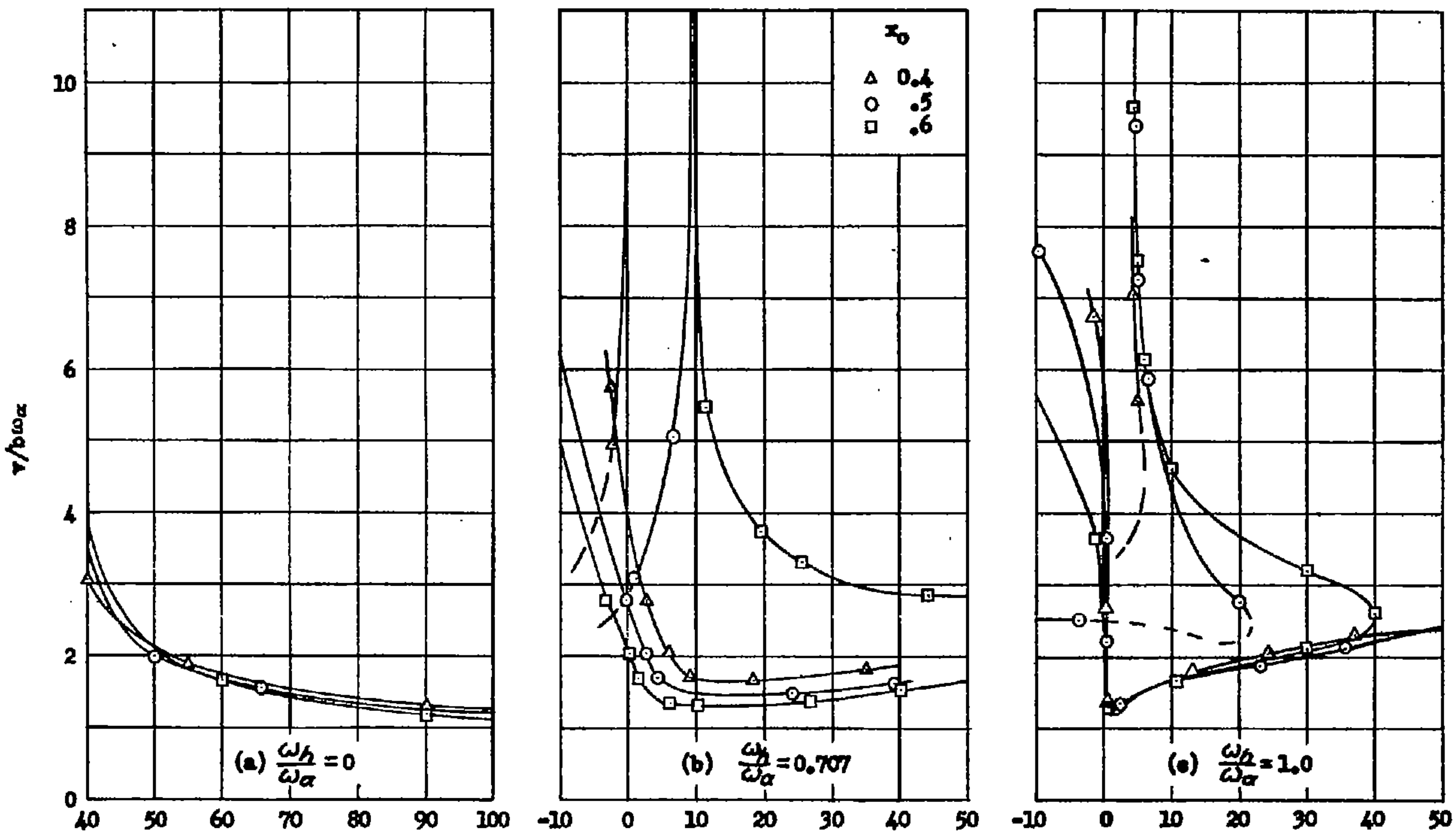
Figure 5.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = \frac{10}{9}$; $\mu = 3.927$.



Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

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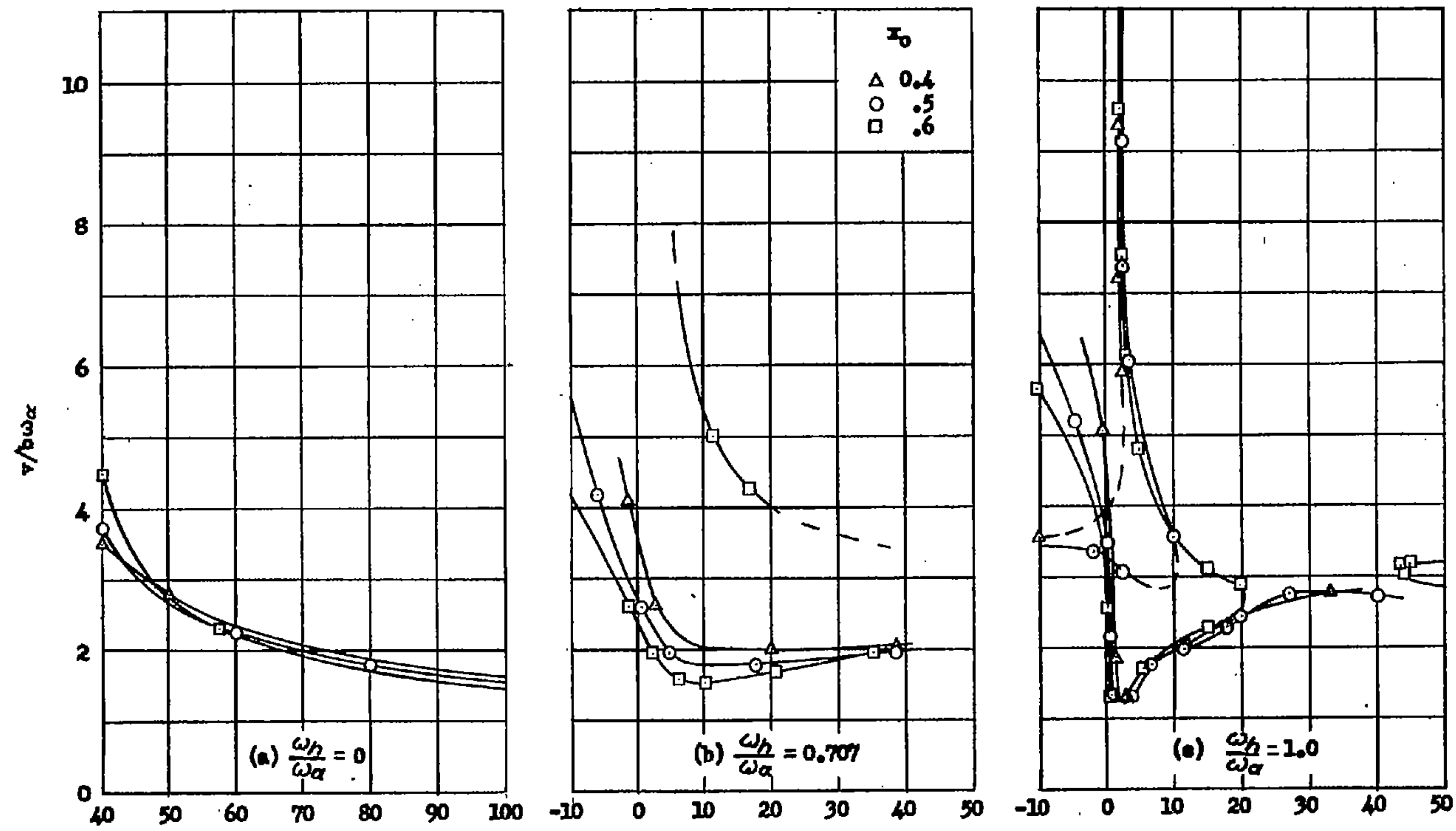
Figure 6.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = \frac{10}{9}$; $\mu = 7.854$.



Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

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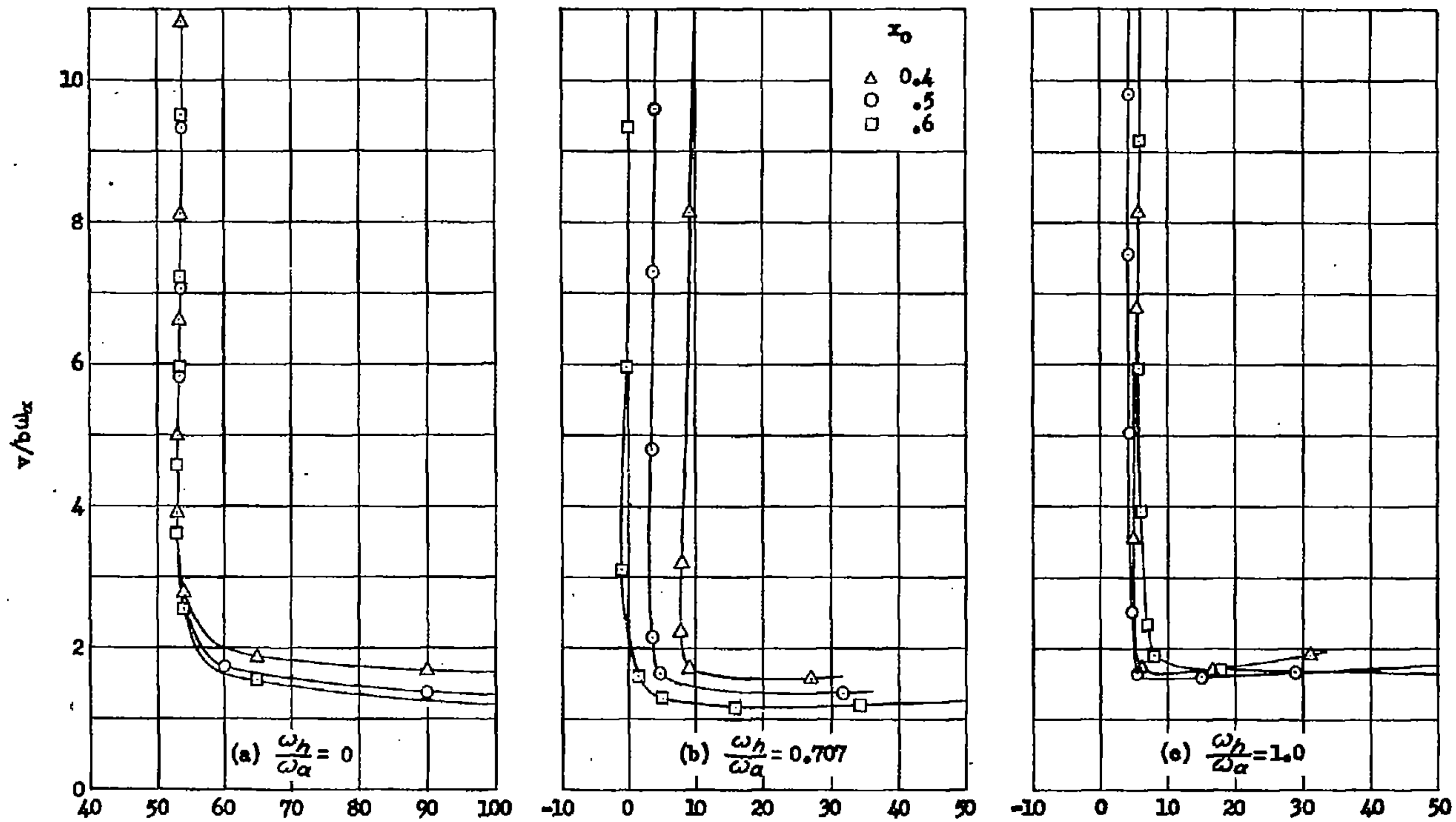
Figure 7.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = \frac{10}{9}$; $\mu = 15.708$.



Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

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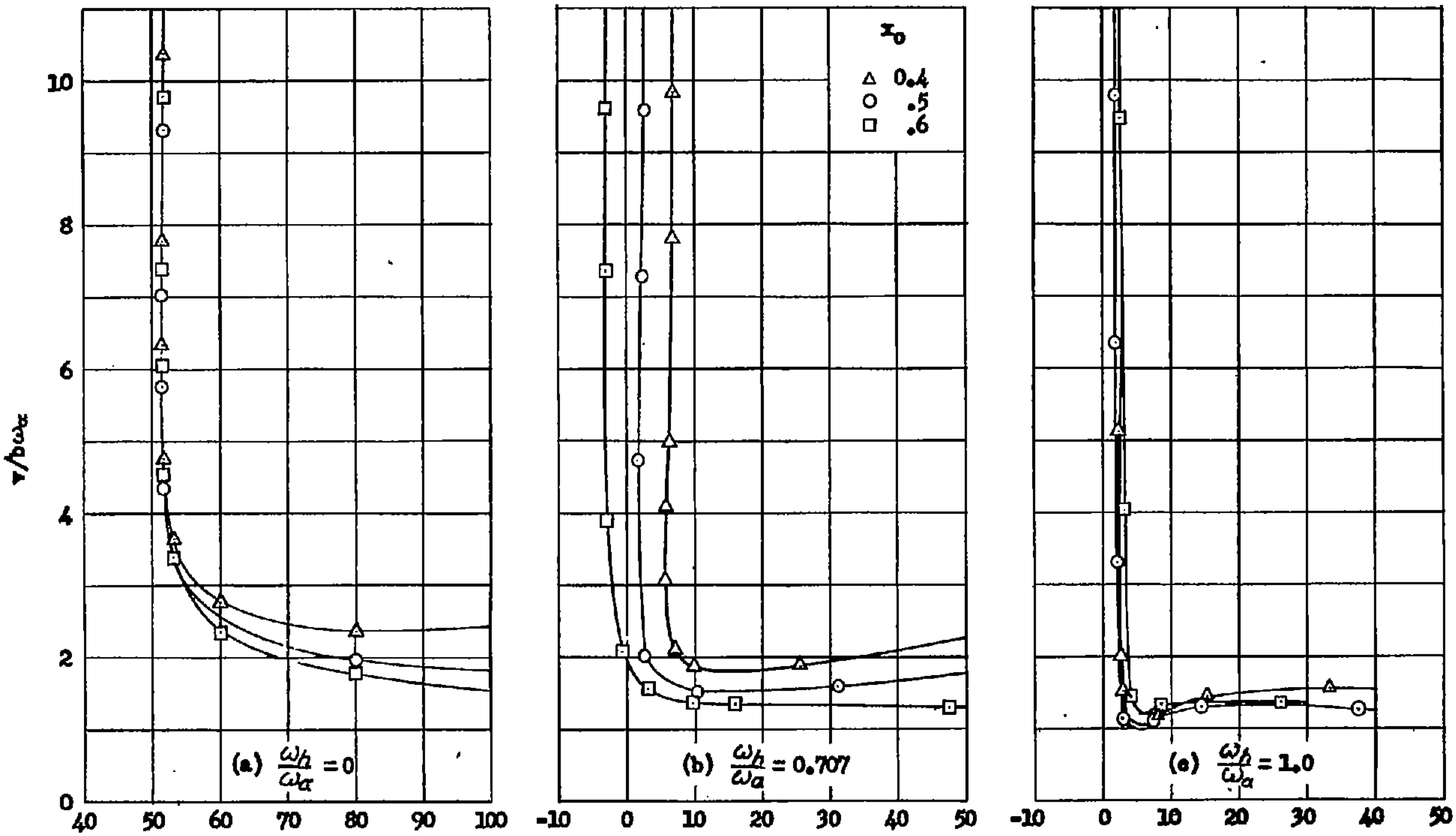
Figure 8.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = \frac{10}{7}$; $\mu = 3.927$.



Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

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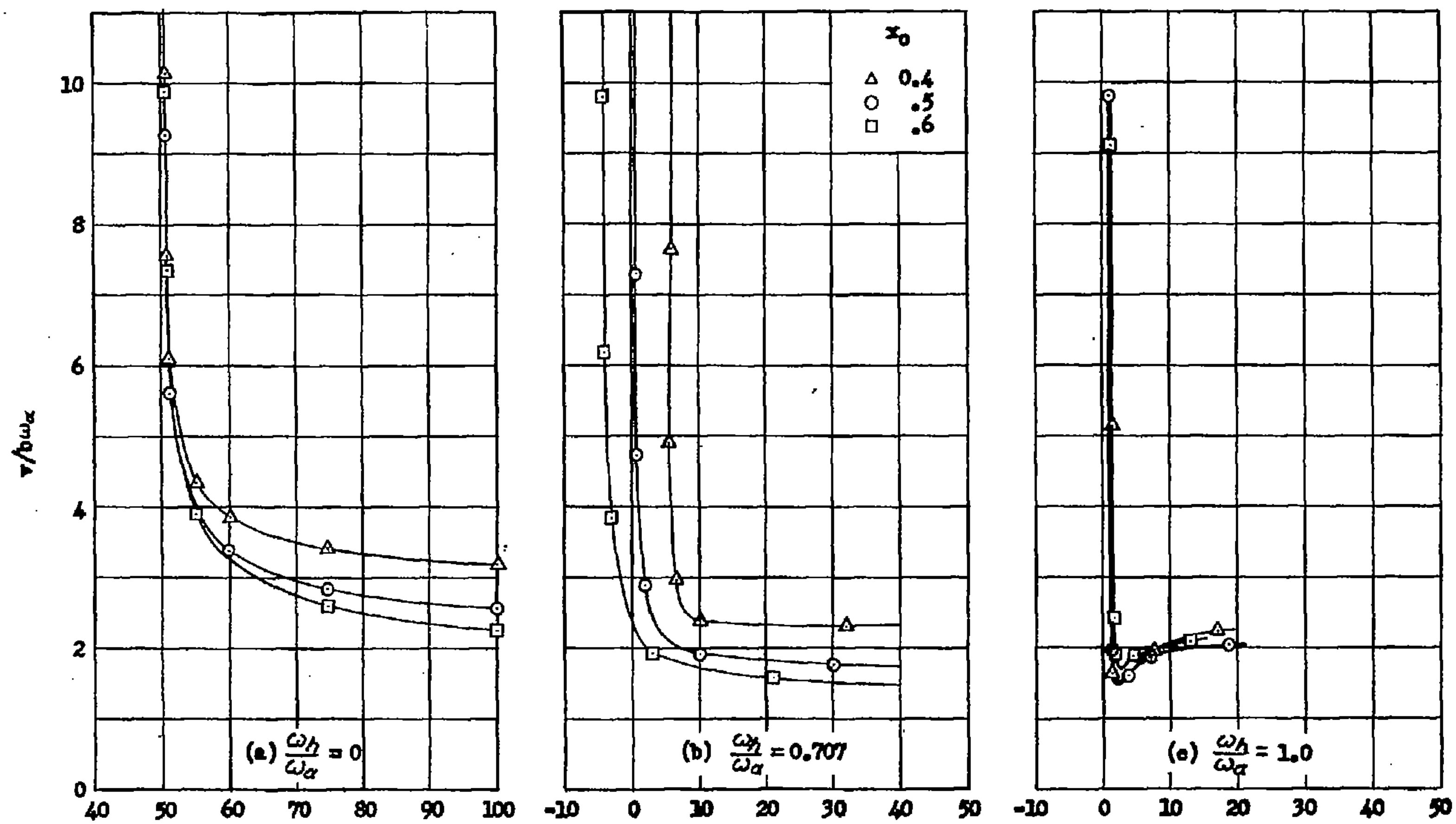
Figure 9.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = \frac{10}{7}$; $\mu = 7.854$.



Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

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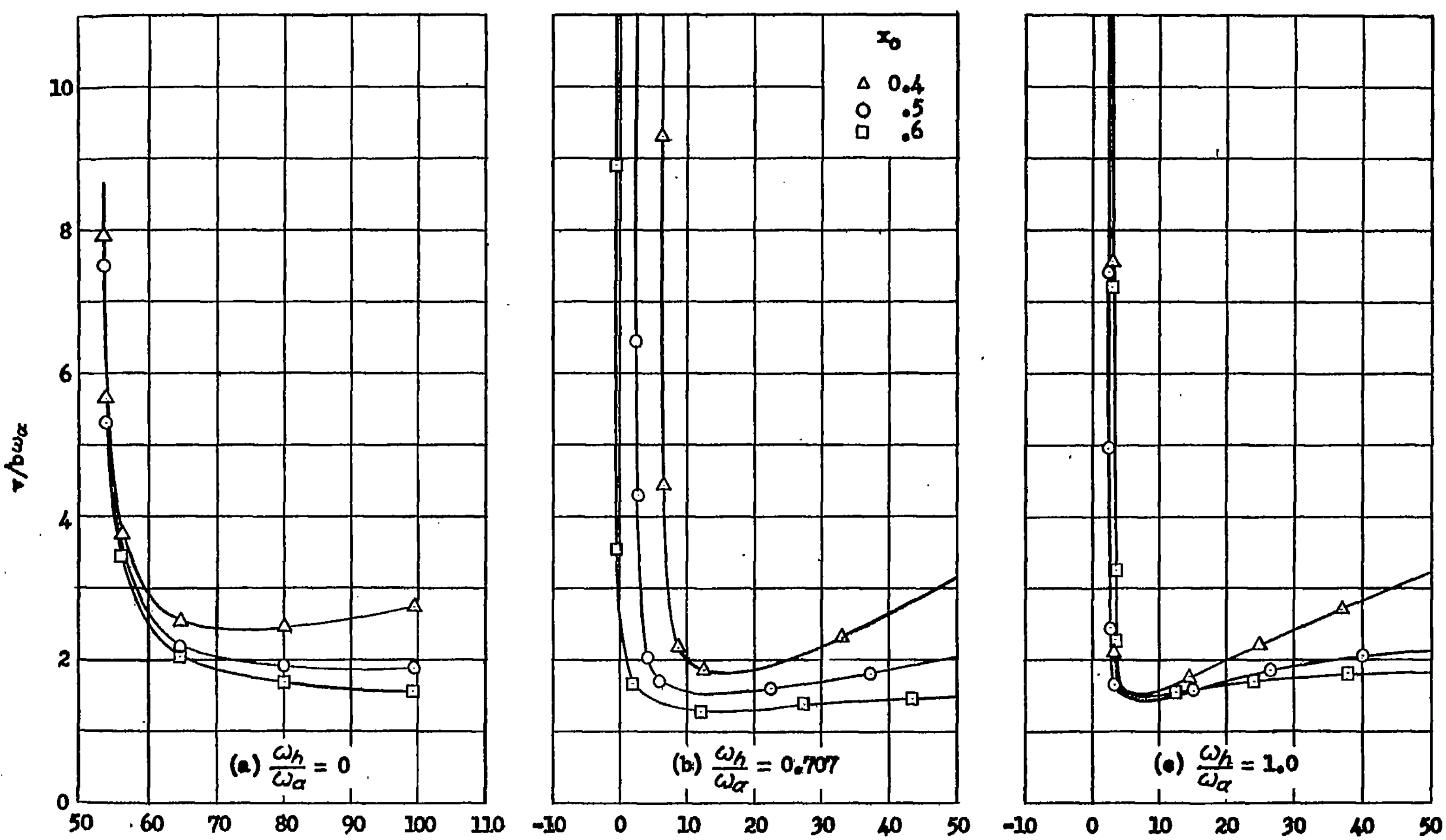
Figure 10.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = \frac{10}{7}$; $\mu = 15.708$.



Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

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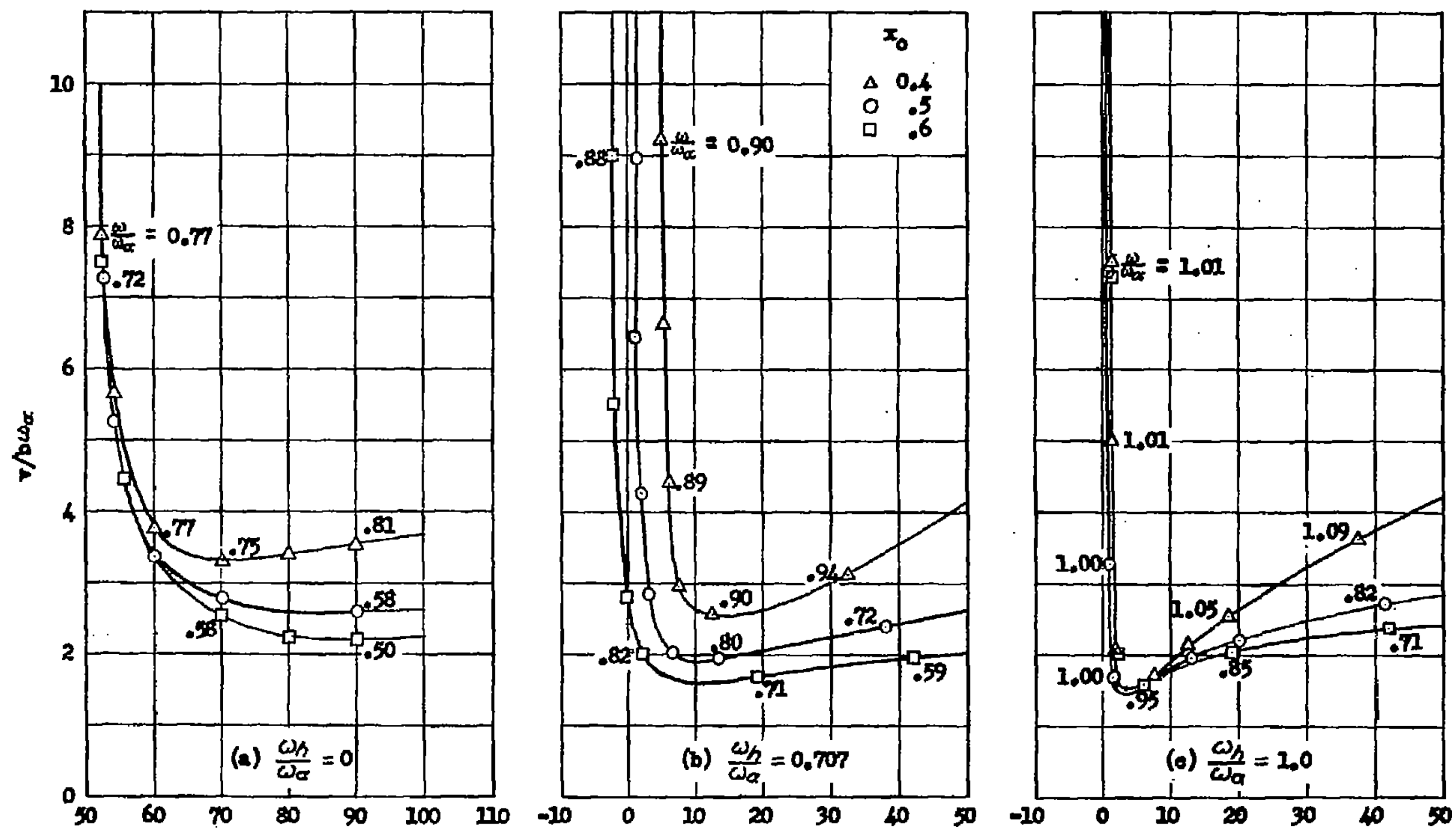
Figure 11.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. M = 2; μ = 3.927.



Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

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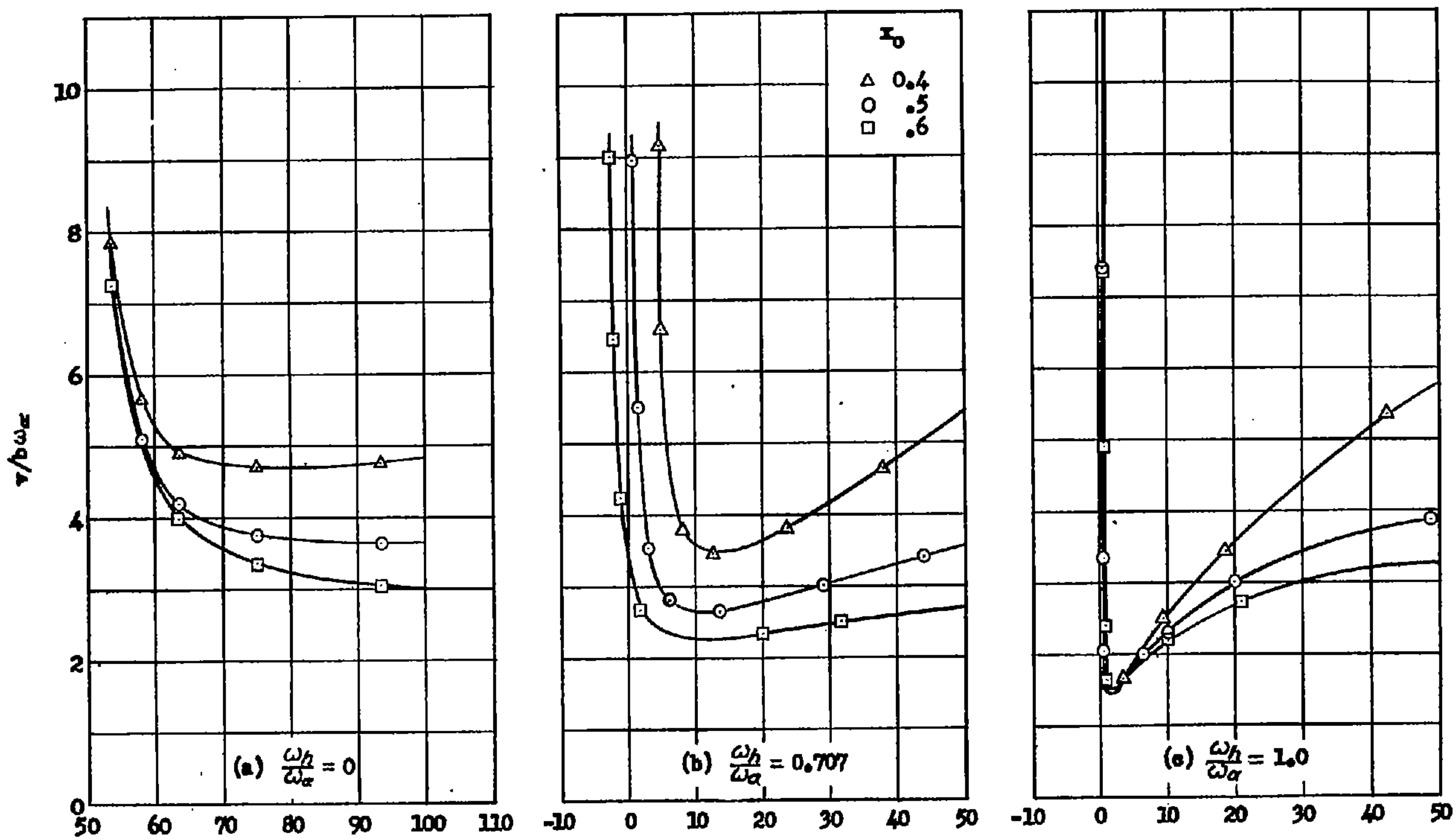
Figure 12.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = 2$; $\mu = 7.854$.



Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

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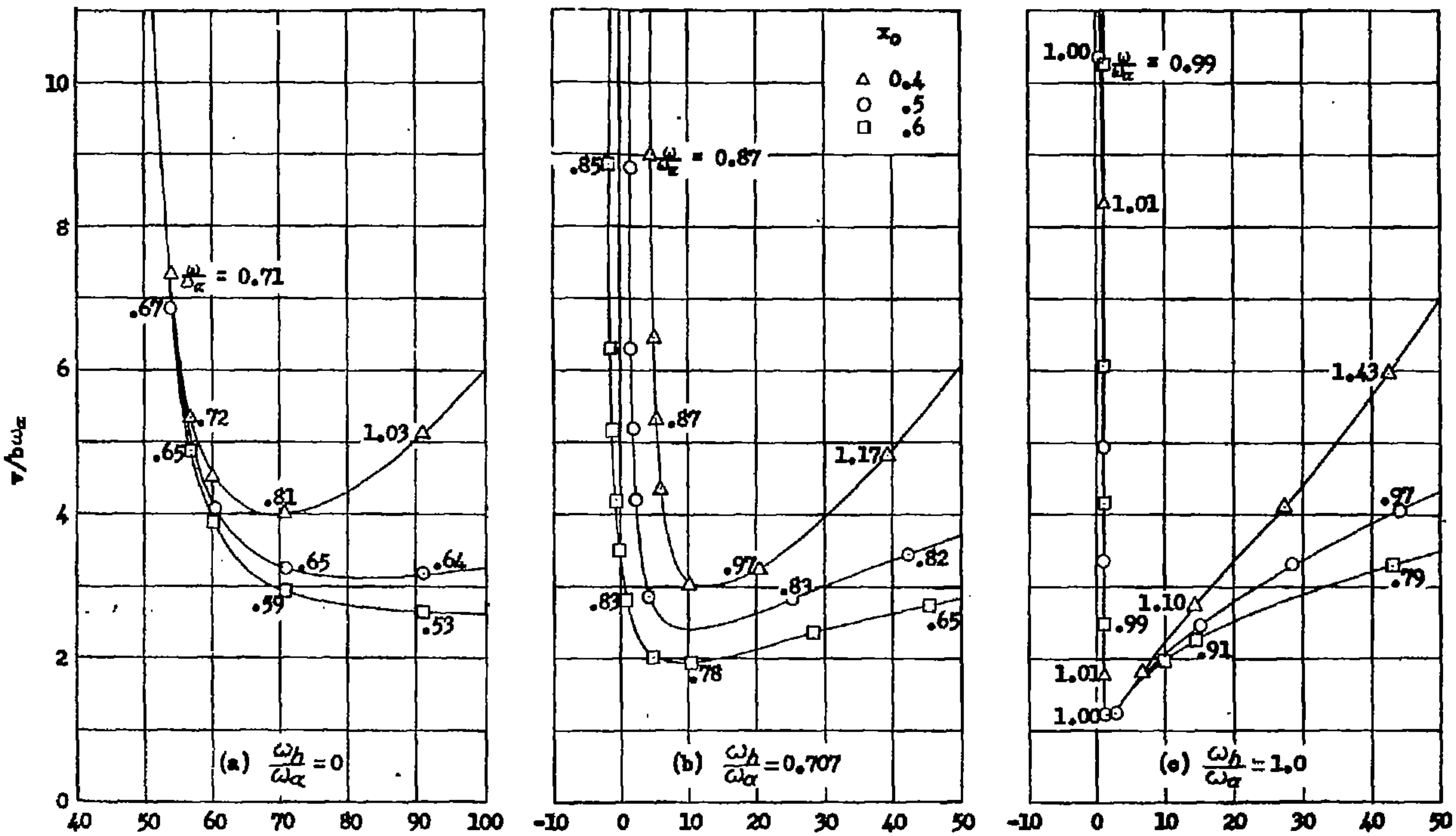
Figure 13.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. $M = 2; \mu = 15.708$.



Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

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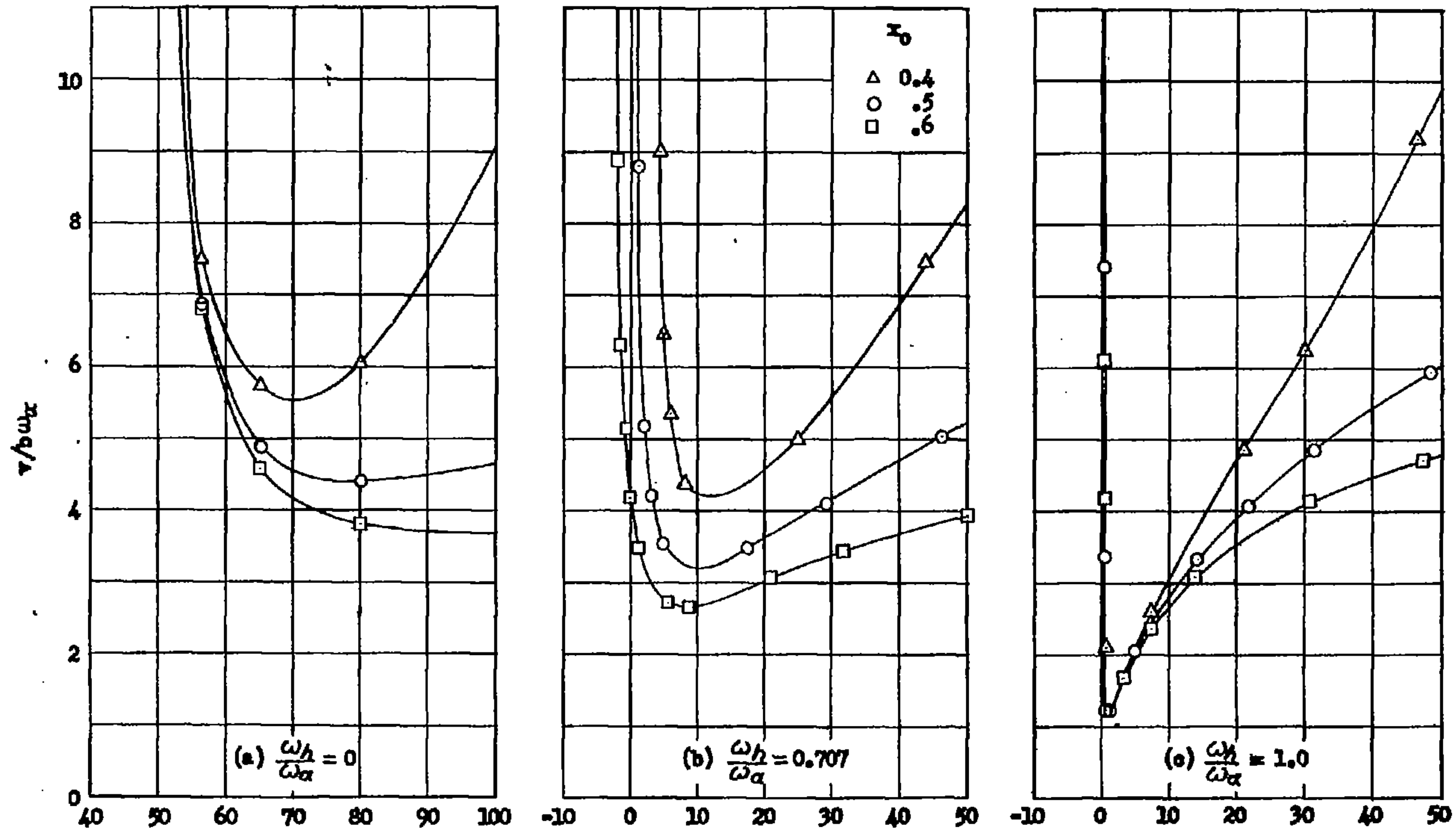
Figure 14.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. M = 5; μ = 3.927.



Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

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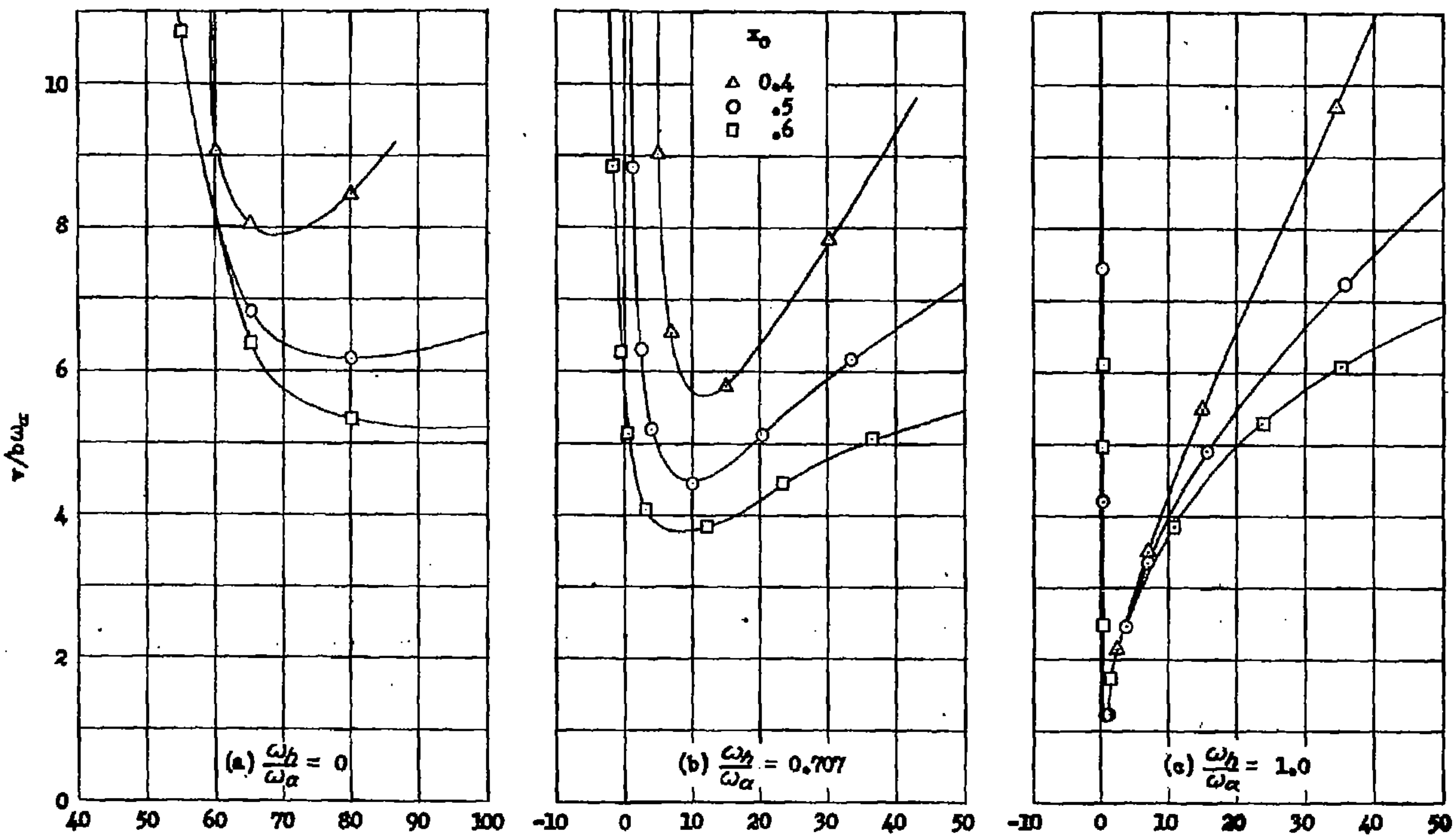
Figure 15.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. M = 5; μ = 7.854.



Center-of-gravity location, percent chord
(a) Measured from leading edge.
(b) and (c) Measured from elastic axis.

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Figure 16.- The flutter coefficient against center-of-gravity location for several positions of elastic axis and for three values of the frequency ratio. M = 5; μ = 15.708.

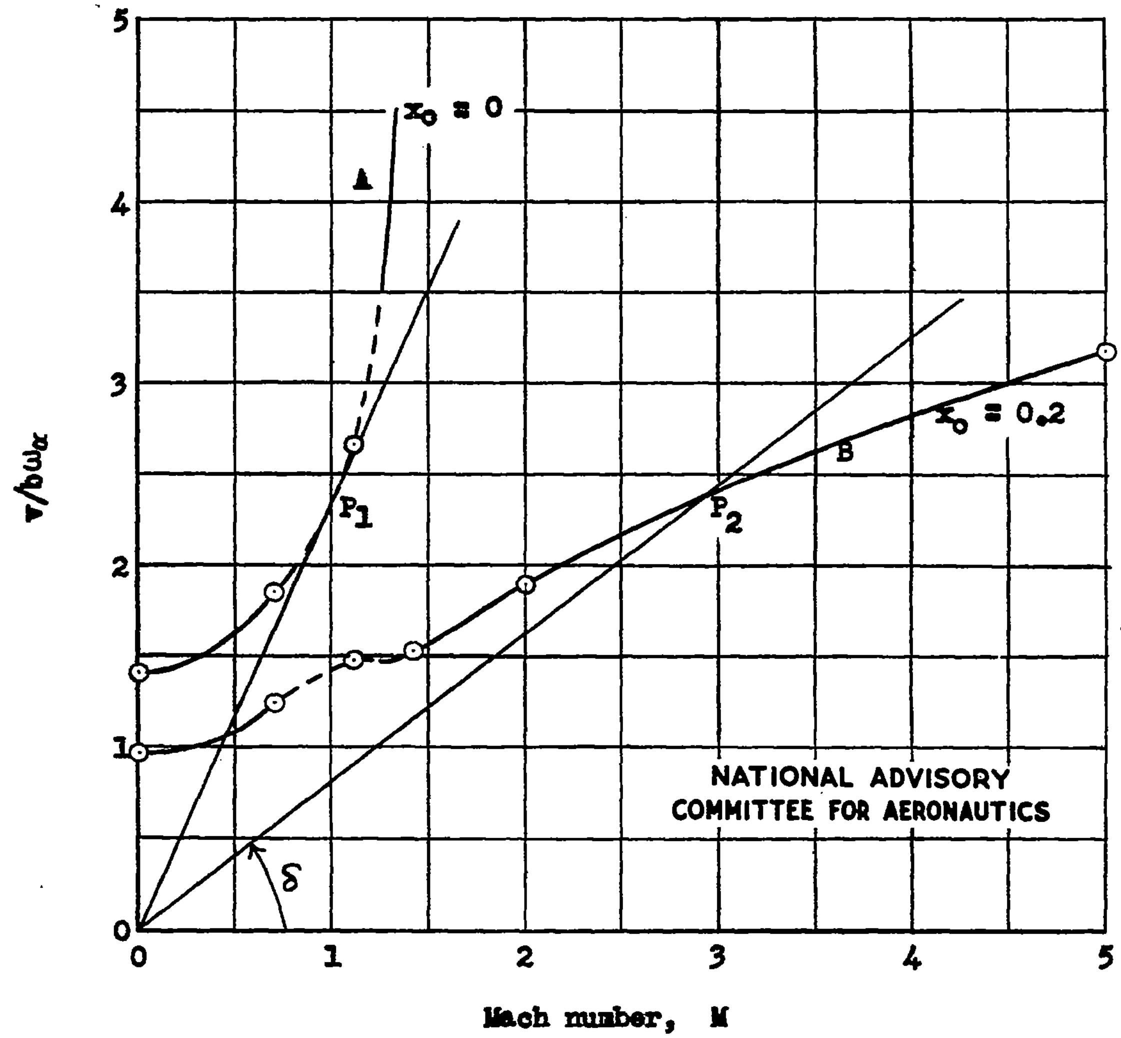

Figure 17.- The flutter coefficient against Mach number for two locations of the center of gravity. Other parameters are $\frac{\omega_h}{\omega_\alpha}$ = 0.707; a = 0; μ = 7.854.

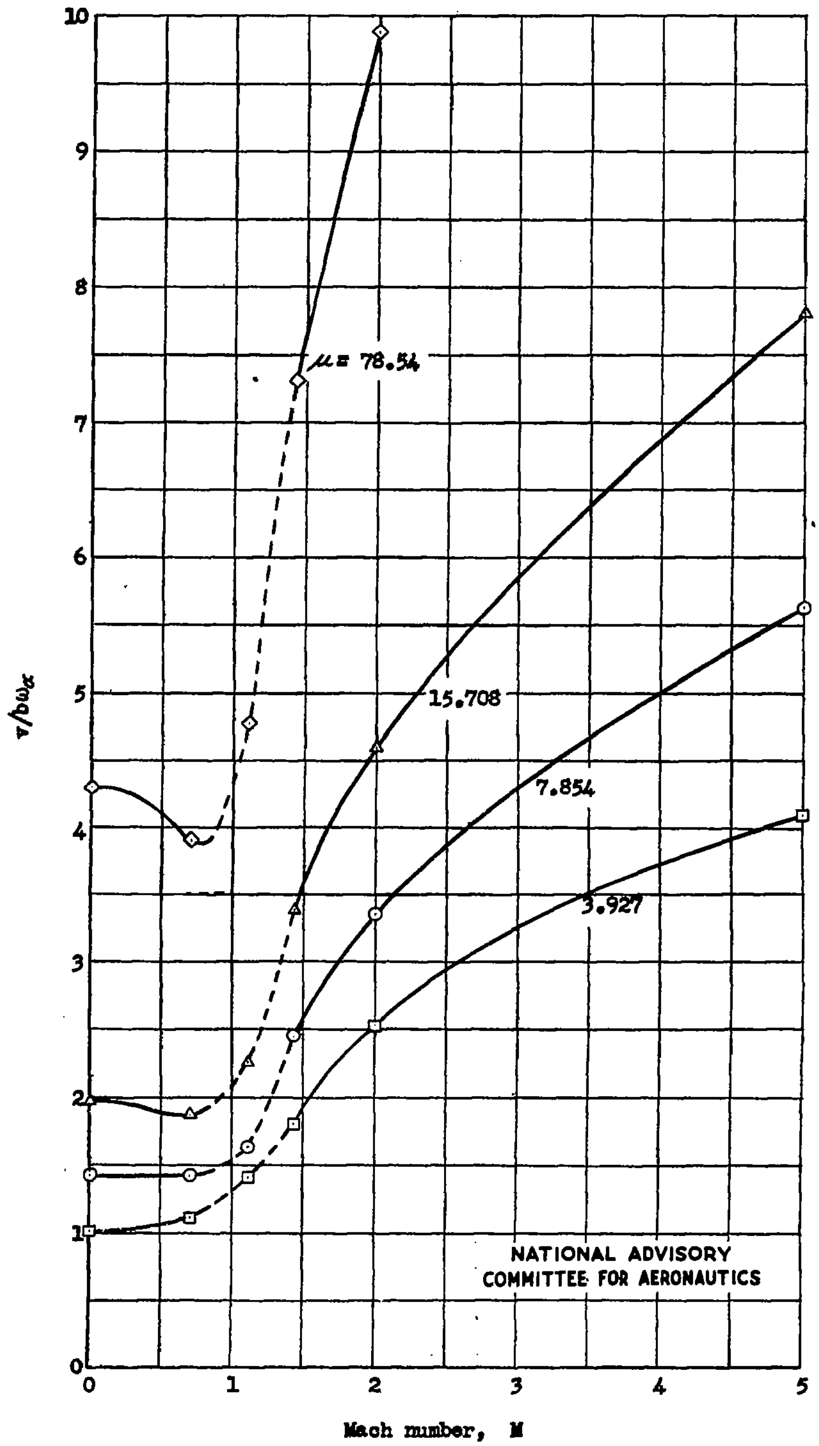

Figure 18.- The flutter coefficient against Mach number for several values of μ. Other parameters are $\frac{\omega_h}{\omega_\alpha} = 0$; $x_\alpha = 0.2$; $a = 0$.

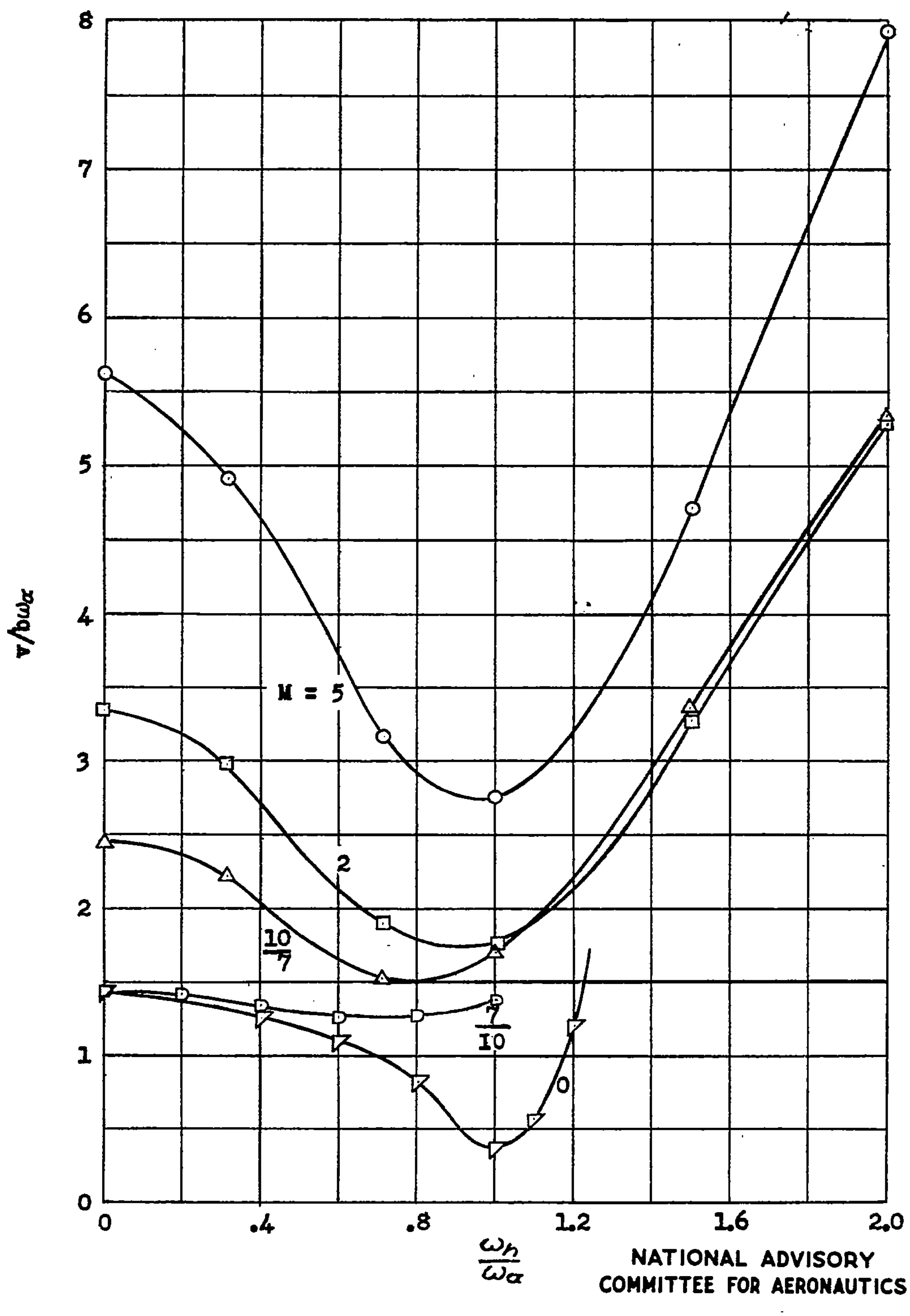

Figure 19.- The flutter coefficient against frequency ratio for several values of M. Other parameters are a = 0; x_α = 0.2; μ = 7.854.

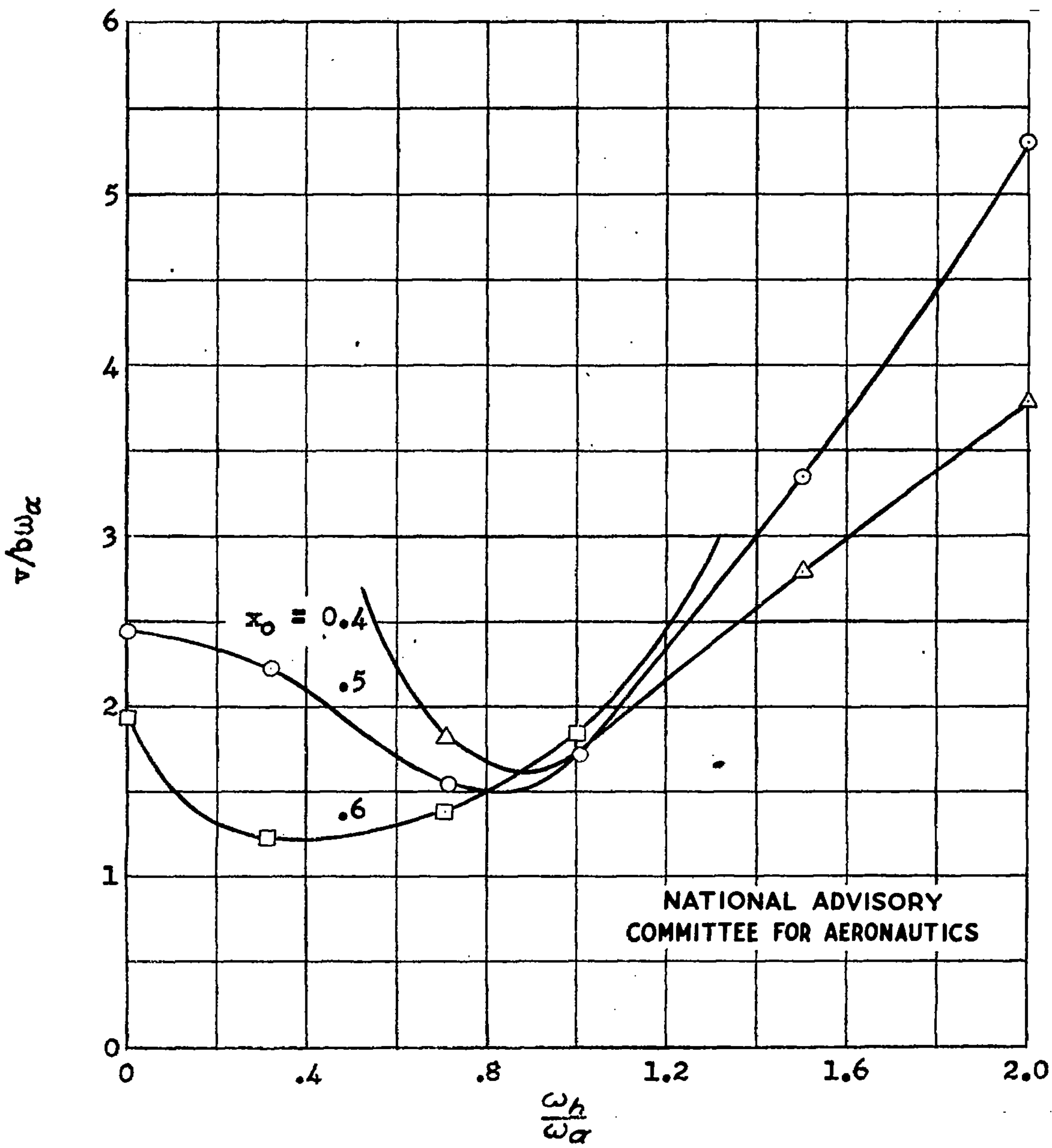

Figure 20.- The flutter coefficient against frequency ratio for three values of x_o. Other parameters are $M = \frac{10}{7}$; $a = 0$; $\mu = 7.854$.

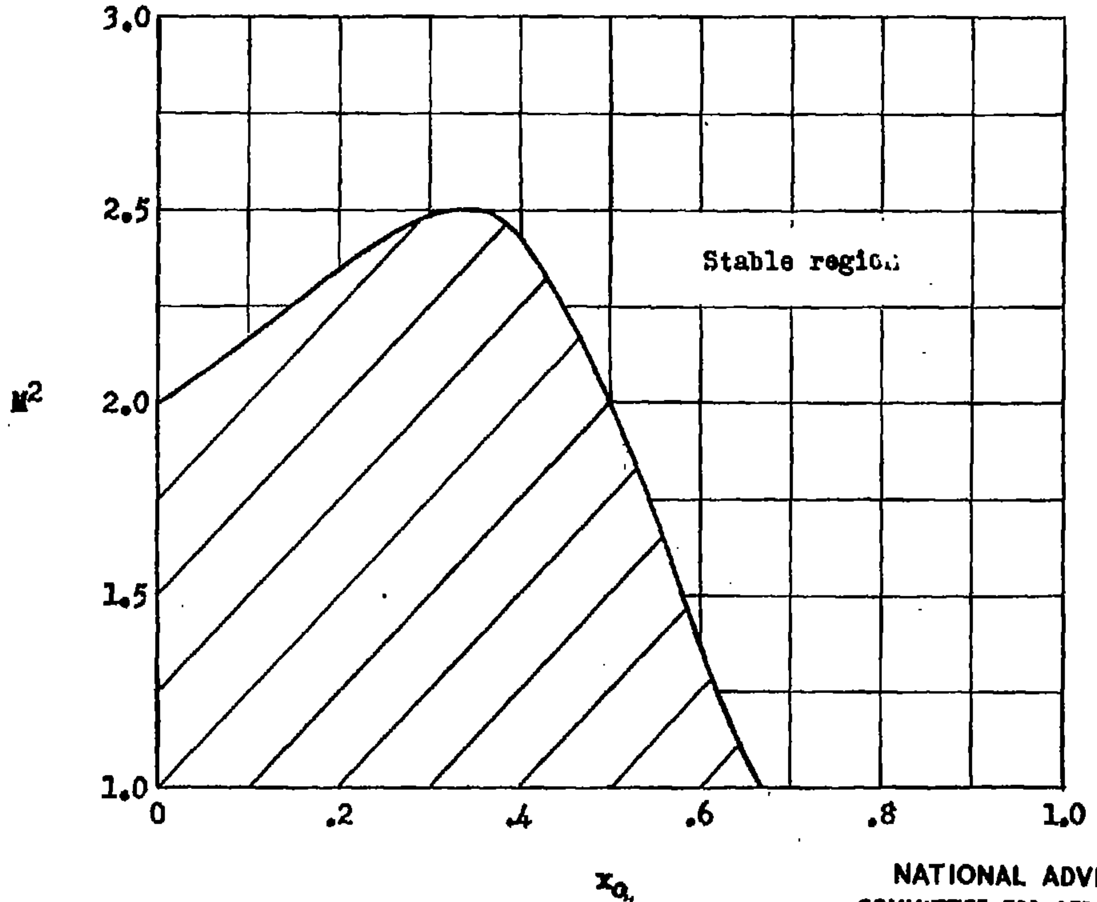

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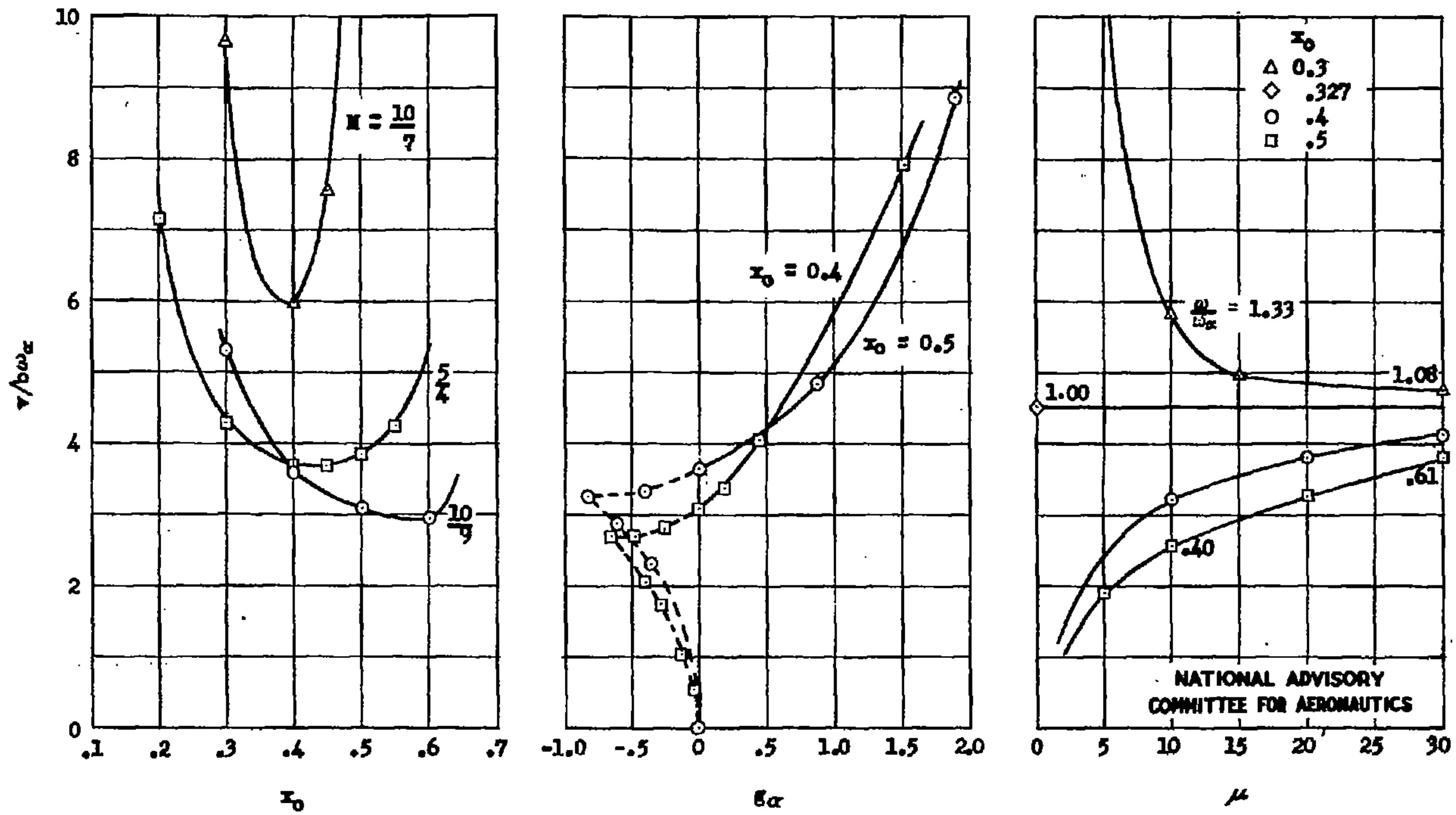
Figure 21.- Plot of $M_4(M,x_o) = 0$.



(a) Flutter coefficient against axis-of-rotation position for several values of M (μ = 15.708). Note that the range of x_o narrows with increase in M and disappears at M = 1.58 and x_o = 0.33.

(b) Flutter coefficient against torsional damping coefficient for two values of x_o ($M = \frac{10}{9}$; μ = 15.708). Negative damping values are shown dashed and have no physical existence.

(c) Flutter coefficient against wing-density parameter μ for several values of x_o ($M = \frac{10}{9}$). The straight-line curve shown corresponds to $M_3 = 0$ (x_o = 0.327).

Figure 22.- Curves for one-degree-of-freedom torsional instability.